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International Conference on
Matrix Analysis and
its Applications

Book of Abstracts

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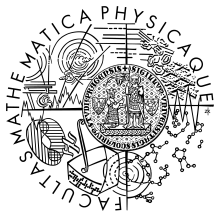


MAT TRIAD 2019

Edited by
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Preface

This volume contains the Book of abstracts of the 8^{th} International Conference on Matrix Analysis and its Applications, MAT TRIAD 2019. The MATTRIAD conferences represent a platform for researchers in a variety of aspects of matrix analysis and its interdisciplinary applications to meet and share interests and ideas. The conference topics include matrix and operator theory and computation, spectral problems, applications of linear algebra in statistics, statistical models, matrices and graphs as well as combinatorial matrix theory and others. The goal of this event is to encourage further growth of matrix analysis research including its possible extension to other fields and domains.

MAT TRIAD 2019 is a registered as

- satellite meeting of ICIAM 2019, The International Congress on Industrial and Applied Mathematics, to be held at Valencia, Spain, July 15-19, 2019, `http://iciam2019.org/`,
- ILAS-Endorsed Meeting, `https://www.ilasic.org/misc/meetings.html`.

MAT TRIAD 2019 in Liblice in Czech Republic is organized based on the successful concept of previous MAT TRIAD editions, which are held biannually since 2005. The conference is scheduled for five days with a series of invited lectures by leading experts in their field.

The invited speakers are:

- Dario Bini (University of Pisa)
- Mirjam Dür (University of Augsburg)
- Arnold Neumaier (University of Vienna)
- Martin Stoll (Technical University of Chemnitz)

Two of the invited speakers serve as lectures:

- Shmuel Friedland (University of Illinois) - the Hans Schneider ILAS Lecturer
- Zdeněk Strakoš (Charles University)

Two recipients of the Young Scientists Awards from MAT TRIAD 2017, which took place in Będlewo, are also invited to give their lecture:

- Álvaro Barreras (Universidad Internacional de La Rioja)
- Ryo Tabata (National Institute of Technology)

There are four special sessions organized:

- *Total positivity*
  organiser: Mohammad Adm, Palestine Polytechnic University, Hebron & Jürgen Garloff, University of Applied Sciences and University of Konstanz
- *Tropical matrix algebra and its applications*
  organiser: Aljoša Peperko, University of Ljubljana, Slovenia & Sergei Sergeev, University of Birmingham, UK
- *Recent developments of verified numerical computations*
  organizer: Takeshi Ogita, Tokyo Woman's Christian University & Siegfried M. Rump, Hamburg University of Technology
- *Interval matrices*
  organiser: Milan Hladík, Charles University, Prague

First, we wish to thank International Linear Algebra Society and RSJ Foundation, to make the conference possible as well as other supporters and sponsors. We thank the members of scientific committee for their work and feedback on the event and to members of local organizing committee for their cooperation and help with necessary organizational tasks. We want to thank all invited speakers for accepting invitations and preparations of their presentations as well as all participants of the conference to make the event possible. We wish them rich time and fruitful discussions during MAT TRIAD 2019 in Liblice.

Miroslav Rozložník, Milan Hladík

The final program is available at `https://mattriad.math.cas.cz/`

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Contents

Invited talks

Tridiagonal inverses of tridiagonal M-matrices and related pentadiagonal matrices

Álvaro Barreras[1] and Juan Manuel Peña[2]

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Keywords: tridiagonal, M-matrix, tridiagonal inverse, pentadiagonal

Abstract

A matrix $A = (a_{ij})_{1\leq i,j\leq n}$ is tridiagonal if $|i-j| > 1$ implies that $a_{ij} = 0$ and it is pentadiagonal if $|i-j| > 2$ implies that $a_{ij} = 0$. Let us recall that a matrix A is a nonsingular M-matrix if it has nonpositive off-diagonal entries and its inverse has nonnegative entries, $A^{-1} \geq 0$ (cf. [4], [2], [3]). Let us also recall that a matrix is called totally positive if all its minors are nonnegative (cf. [1]).

Imam provided some partial results on tridiagonal matrices whose inverse is M-matrix in [5]. On the other hand, the inverse of a nonsingular tridiagonal M-matrix is analyzed in [6].

In this talk, a necessary and sufficient condition in order to guarantee that the inverse of a tridiagonal M-matrix is tridiagonal is provided. This condition is provided in terms of the zero pattern of A. A sufficient condition in order to assure that the inverse of a nonsingular totally positive matrix is a nonsingular M-matrix is also presented by using the mentioned result.

Then, we provide a necessary condition in order to guarantee that the inverse of a pentadiaonal M-matrix is also pentadiagonal.

Finally, we include some examples that illustrate the necessity of the conditions for that results.

Acknowledgement

This work has been partially supported by the Spanish Research Grant MTM2015-65433-P (MINECO/FEDER), by Gobierno de Aragón and Fondo Social Europeo.

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Solving matrix equations encountered in stochastic processes: an algorithmic analysis

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Keywords: matrix equations, infinite matrices, Toeplitz matrices, queueing models, Markov chains

Abstract

Many queuing problems are modeled by Markov chains with infinitely many states. The computation of the invariant probability measure, performed by means of matrix-geometric techniques [7], relies on the solution of matrix equations expressed in terms of a matrix polynomial or of a matrix power series like, for instance, in the QBD and in the M/G/1 processes [3], [4], [7]. Moreover, in several important cases from the applications, the coefficients of these matrix polynomials / power series are infinite matrices having a Toeplitz-like structure [5], [6]. The demand from applications is to provide solution algorithms which are highly effective both in terms of low computational cost and of numerical reliability and which can manage with the infinite nature of the problem.

In this talk we provide an overview of this kind of equations and their motivations, then we address some theoretical and computational issues encountered in the analysis of this class of problems.

This work continues the analysis performed in [1], [2].

Acknowledgement

This work has been supported by GNCS of INdAM.

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Copositive optimization and completely positive matrix factorization

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Keywords: copositive and completely positive matrices, factorization of matrices

A copositive optimization problem is a problem in matrix variables with a constraint which requires that the matrix be in the cone of copositive symmetric $n \times n$ matrices. This cone is defined as

$$\mathcal{COP}_n := \{A \in \mathbb{R}^{n\times n} \mid A = A^T, \text{ and } x^T Ax \geq 0 \text{ for all } x \geq 0\}.$$

Its dual cone which appears in the dual optimization problem is the cone of completely positive matrices:

$$\begin{aligned}\mathcal{CP}_n &:= \{A \in \mathbb{R}^{n\times n} \mid A = BB^T \text{ where } B \in \mathbb{R}^{n\times r}, B \geq 0\} \\ &= \text{conv}\{xx^T \mid x \in \mathbb{R}^n_+\}.\end{aligned}$$

These cones have received considerable attention in the mathematical optimization community because it has turned out that many non-convex quadratic optimization problems can be formulated as linear problems over these cones. This was first shown by Bomze et al. [1] for the so-called standard quadratic optimization problem of finding the minimum of a (not necessarily convex) quadratic function over the standard simplex: Let $Q \in \mathbb{R}^{n\times n}$ be a symmetric matrix, let $e \in \mathbb{R}^n$ denote the all-ones vector, and consider the problem

$$\begin{array}{ll}\min & x^T Qx \\ \text{st.} & e^T x = 1, \\ & x \in \mathbb{R}^n_+.\end{array} \qquad \text{(StQP)}$$

It was shown by Bomze et al. [1] that (StQP) can be equivalently formulated as

$$\begin{array}{ll} \min & \langle Q, X\rangle \\ \text{st.} & \langle E, X\rangle = 1, \\ & X \in \mathcal{CP}_n \end{array} \tag{1}$$

where $E = ee^T$ denotes the all ones matrix in $\mathbb{R}^{n\times n}$. The dual of (1) is the copositive optimization problem

$$\begin{array}{ll} \max & y \\ \text{st.} & Q - yE \in \mathcal{COP}_n, \\ & y \in \mathbb{R}, \end{array} \tag{2}$$

and it can be shown that the optimal values of (1) and (2) are equal.

The reformulation of (StQP) as (1) or (2) is remarkable since by this approach, a nonconvex optimization problem is reformulated equivalently as a convex problem. The complexity of the original problem is entirely shifted into the cone constraint. The approach was later extended and it was shown that many other nonconvex quadratic optimization problems and also many combinatorial problems like the maximum clique problem and others can be reformulated as linear optimization problems over $\mathcal{CP}_n$ or $\mathcal{COP}_n$. More details can be found in [2].

The equivalence between (StQP) and problem (1) is understood in the following sense: if x^* is an optimal solution of (StQP), then the matrix $x^*(x^*)^T$ is an optimal solution of (1). Conversely, if X^* is an optimal solution of (1), then we can write $X^* = \sum_{i=1}^k x_i x_i^T$ with $x_i \in \mathbb{R}^n_+$ for all i, and an appropriately scaled multiple of each x_i is then an optimal solution of (StQP).

For this reason, it is necessary to be able to factorize a given $X^* \in \mathcal{CP}_n$ as $X^* = \sum_{i=1}^k x_i x_i^T$ with $x_i \in \mathbb{R}^n_+$ for all i. In [3], a method was proposed to solve this factorization problem: we reformulated the factorization problem as a nonconvex feasibility problem and developed a solution method based on alternating projections. A local convergence result can be shown for this algorithm, and numerical experiments show that the algorithm performs very well in practice.

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The Collatz-Wielandt quotient for pairs of nonnegative operators

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Keywords: Perron-Frobenius theory, Collatz-Wielandt quotient, completely positive operators, commodity pricing, wireless networks, quantum information theory.

Abstract

In these two lectures we consider the Collatz-Wielandt quotient for a pair of nonnegative operators A, B that map a given pointed generating cone in the first space into a given pointed generating cone in the second space [3]. In the case the two spaces and the two cones are identical, and B is the identity operator this quotient is the spectral radius of A. In some applications, as commodity pricing, power control in wireless networks and quantum information theory, one needs to deal with the Collatz-Wielandt quotient for two nonnegative operators. In this paper we treat the two important cases: a pair of rectangular nonnegative matrices and a pair completely positive operators. We give a characterization of minimal optimal solutions and polynomially computable bounds on the Collatz-Wielandt quotient.

Outline of the two lectures

In the first lecture and the beginning of the second lecture we will cover the following topics: We will define the Collatz-Wielandt type infmax problem for a pair of nonnegative operators $A, B : \mathbb{R}^{N_1} \to \mathbb{R}^{N_2}$,

with respect to closed pointed generating cones $\mathbf{K}_i \subset \mathbb{R}^{N_i}$ for $i = 1, 2$: $A\mathbf{K}_1, B\mathbf{K}_1 \subseteq \mathbf{K}_2$. Denote by $\mathbf{K}_i^o$ the interior of $\mathbf{K}_i$. Let

$$r(A, B, x) = \inf\{t, t \in [0, \infty], tBx - Ax \in \mathbf{K}_2\} \quad \text{for } x \in \mathbf{K}_1 \setminus \{0\}. \tag{1}$$

$$\rho(A, B) = \inf\{r(A, B, x), x \in \mathbf{K}_1^o\}. \tag{2}$$

In general, $\rho(A, B)$ can have any value in $[0, \infty]$. We call $\rho(A, B)$ the *Collatz-Wielandt quotient*.

We will discuss the Collatz-Wielandt quotient for a pair of rectangular nonnegative matrices $A, B \in \mathbb{R}_+^{m \times n}$, i.e. :

$$\rho(A, B) := \inf_{x=(x_1,\ldots,x_n)^\top > 0} \max_{i \in [m]} \frac{(Ax)_i}{(Bx)_i}. \tag{3}$$

We will give characterizations of the optimal x and the polynomial computability of $\rho(A, B)$ in this case. We will bring an application to the wireless networks as discussed in [1], [2].

In the second part of lecture two we will discuss Collatz-Wielandt quotient of pairs of completely positive operators, which are frequently appear in quantum information theory as quantum channels. These are special positive operators on the cone of positive semidefinite hermitian matrices.

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Confidence intervals for large-scale linear least squares solutions

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Keywords: iterative methods, estimation, confidence regions

Confidence intervals for the solutions of linear least squares problems are traditionally given in terms of the inverse of the coefficient matrix of the normal equations.

In the large-scale case it is not feasible to compute the inverse explicitly. We show how one can nevertheless compute meaningful approximations to these confidence intervals.

From PDEs to data science: an adventure with the graph Laplacian

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In this talk we briefly review some basic PDE models that are used to model phase separation in materials science. They have since become important tools in image processing and over the last years semi-supervised learning strategies could be implemented with these PDEs at the core. The main ingredient is the graph Laplacian that stems from a graph representation of the data. This matrix is large and typically dense. We illustrate some of its crucial features and show how to efficiently work with the graph Laplacian. In particular, we need some of its eigenvectors and for this the Lanczos process needs to be implemented efficiently. Here, we suggest the use of the NFFT method for evaluating the matrix vector products without even fully constructing the matrix. We illustrate the performance on several examples.

Operator preconditioning, spectral information and convergence behavior of Krylov subspace methods

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Keywords: operator preconditioning, Krylov subspace methods, spectral decomposition, infinite-dimensional analysis, finite-dimensional computations, nonlinearity, effects of rounding errors

Introduction

Krylov subspace methods such as the method of conjugate gradients (CG) were developed as tools for solving linear problems with finite matrices but almost immediately they were also reformulated for operators at infinite dimensional Hilbert spaces. They are important as computational methods. They are equally important as fascinating mathematical objects linked with classical approximation theory and functional analysis that provoke questions going across the fields [2].

The structure of lectures

This contribution will combine historical perspectives with some recent developments concerning the role of the spectral information in the numerical solution of elliptic PDEs. The first lecture will attempt to use the infinite dimensional view in parallel with the standard finite dimensional matrix descriptions to show similarities, differences, subtleties and consequences that affect developments in the area. The

second lecture will examine arguments used in analysis and in justification of preconditioning techniques in practical computations, including the effects of rounding errors [1], [3].

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Immanants and symmetric functions

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Keywords: immanants, invariant matrices, Schur functions

Introduction

An immanant is a matrix function associated with the representation of the symmetric group. The permanent and determinant are the special cases where the trivial and alternating representations are applied respectively.

The theory of the symmetric functions has been developed with the background of the representation theory. One of the most important basis for the space of symmetric functions consists of Schur functions, indexed by partitions of positive integers or Young diagrams. The product of two Schur functions is described in a combinatorial way, called the Littlewood-Richardson rule. In [3], it is shown that immanants also have expansions in terms of matrix minors in the same rule.

In this talk, we review the classical inequality problem about immanants which originates from Schur [4]. The permanental dominance conjecture [1], an analogue of Schur's result, is still open, although much work on it was done in 1970–1980's. We suggest an approach to sharper bounds for these inequalities. We discuss some limiting behavior of immanants and its related topics.

We also consider immanant identities corresponding to another type of multiplication of Schur functions defined through invariant matrices. Irreducible invariant matrices are also indexed by Young

diagrams, and are generalizations of induced and compound matrices. We observe the contribution of immanants to the representations of the general linear group based on Littlewood's work [2] and some combinatorial aspects such as the Littlewood-Richardson rule.

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Special Sessions

SPECIAL SESSION

Total positivity

Organisers: Mohammad Adm[1], Jürgen Garloff[2]

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The concept of total positivity is rooted in classical mathematics where it can be traced back to works of Schoenberg on variation diminishing properties and of Gantmacher and Krein on small oscillations of mechanical systems. Since then the class of totally positive matrices and operators proved to be relevant in such a wide range of applications that over the years many distinct approaches to total positivity, amenable to a particular notion, have arisen and advocated by many prominent mathematicians. This area is, however, not just a historically significant subject in mathematics, but the one that continues to produce important advances and spawn worth-wile applications. This is reflected by the topics which will be covered by the speakers of the Special Session, viz. the study of classes of matrices related to total positivity and more generally, to sign regularity, accurate computations based on bidiagonalization, inverse eigenvalue problems, and the location of the roots of polynomials.

On the spectral properties of nonsingular matrices that are strictly sign-regular for some order with applications to totally positive discrete-time systems

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Keywords: Totally positive matrix, totally positive differential system, cooperative dynamical system, cyclic sign variation diminishing property, compound matrix.

Introduction

A matrix is called strictly sign-regular of order k (denoted by SSR_k) if all its $k \times k$ minors are non-zero and have the same sign. For example, totally positive matrices, i.e., matrices with all minors positive, are SSR_k for all k, see, e.g., [2],[3]. Another important subclass are those matrices that are SSR_k for all odd k which appear in totally positive discrete-time systems.

Basic properties

Matrices that are SSR_k for all odd k have interesting sign variation diminishing properties, and it has been recently shown in [4] that they play an important role in the analysis of certain nonlinear cooperative

dynamical systems. In fact, the spectral properties of such matrices are not studied before.

Main results

In this talk, the spectral properties of nonsingular matrices that are SSR_k for a specific value k are presented. One of the results is that the product of the first k eigenvalues is real and of the same sign as the $k \times k$ minors, and that linear combinations of certain eigenvectors have specific sign patterns. It is then shown how known results for matrices that are SSR_k for several values of k can be derived from these spectral properties. Using these theoretical results, the notion of a totally positive discrete-time system (TPDTS) is introduced. This may be regarded as the discrete-time analogue of the important notion of a totally positive differential system. It is shown that TPDTSs entrain to periodic excitations.

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An inverse eigenvalue problem for pentadiagonal oscillatory matrices

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Keywords: inverse eigenvalue problem, oscillatory matrix, orthogonal polynomial, Laurent-Jacobi matrix

Introduction

Solving inverse eigenvalue problems (IEPs) is an important subject in numerical linear algebra. Chu and Golub gave a comprehensive review for IEPs [1]. The problem to construct a matrix having prescribed eigenvalues is one of IEPs and called the structured IEP. A matrix A whose minors are all nonnnegative is called totally nonnegative (TN), and an irreducible and invertible TN matrix an oscillatory matrix [2]. In this talk, we consider an IEP for oscillatory matrices with a specific shape and characteristics from the view point of orthogonal polynomials.

Main result

We consider Laurent biorthogonal polynomials (LBPs) $P_n^{(t)}(z)$ with respect to moments $f_i^{(t)} = f_{t+i}$ that satisfy a three-term recursion relation [4]

$$P_0^{(t)}(z) = 1, \quad P_{n+1}^{(t)}(z) = (z - q_n^{(t)})P_n^{(t)} - e_n^{(t)} z P_n^{(t)}(z), \quad n = 0, 1, \ldots. \tag{1}$$

From (1), we obtain a relation among coefficients

$$q_n^{(t+1)} e_n^{(t)} = q_{n-1}^{(t)} e_n^{(t+1)}, \quad q_n^{(t+1)} + e_n^{(t+1)} = q_n^{(t)} + e_{n+1}^{(t)}. \tag{2}$$

By using the equivalence between LBPs and orthgonal Laurent polynomials (OLPs) [3], we can associate (2) with an eigenproblem for a pentadiagonal matrix $A^{(t)}$ called *Laurent-Jacobi matrix*.

Our main result is as follows: Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be positive distinct. Let $c_1, c_2, \ldots, c_m$ be appropriate nonzero constants. We determine a sequence $\{f_t\}_{t=0,1,\ldots}$ by $f_t = c_1\lambda_1^t + c_2\lambda_2^t + \cdots + c_m\lambda_m^t$. Then, we can construct an oscillatory Lanrent-Jacobi matrix $A^{(t)}$ with prescriced eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$ by using (2) with $e_0^{(t)} \equiv 0$ and $q_1^{(t)} = f_{t+1}/f_t$.

Acknowledgement

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Relaxing the nonsingularity assumption for intervals of totally nonnegative matrices

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Keywords: Matrix interval, checkerboard partial order, totally nonnegative matrix, Cauchon matrix, Cauchon algorithm.

Totally nonnegative matrices, i.e., matrices having all their minors nonnegative, and matrix intervals with respect to the checkerboard partial order are considered. In [1] it is proven that if the two bound matrices of such a matrix interval are totally nonnegative and nonsingular, then all matrices from this interval are also totally nonnegative and nonsingular. In our talk, we relax the nonsingularity assumption by assuming the linear independence of certain rows and columns of the two bound matrices.

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Total positivity preservers

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Keywords: totally positive matrix, totally nonnegative matrix, Hankel matrix, entrywise preserver.

Abstract

We classify the entrywise functions that preserve various sets of totally positive matrices of a fixed dimension. These include (a) rectangular matrices, (b) symmetric matrices, and (c) Hankel matrices. The last of these sets (c) is a cone and we explain why its endomorphisms are closely connected to those of positive semidefiniteness, with precise classifications. The first two sets (a), (b) are not cones, and the only functions that preserve them are necessarily powers. In particular, the dimension-free preservers of total positivity are precisely the dilations. (Based on joint works with Alexander Belton, Dominique Guillot, and Mihai Putinar; and with Terence Tao.)

Acknowledgements

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Accurate eigenvalues and zero Jordan blocks of (singular) sign-regular matrices

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Keywords: accuracy, eigenvalues, sign-regular matrices

Introduction

We present new algorithms for computing all eigenvalues and *zero Jordan blocks* (i.e., ones corresponding to eigenvalue 0) of sign-regular matrices of signature $\{1, -1, 1, -1, \ldots\}$ to high relative accuracy in floating point arithmetic. These matrices can also be described as *totally nonnegative* with columns in reversed order. These results generalize our earlier ones for the nonsingular case [2].

Definitions

Matrices with all minors nonnegative are called totally nonnegative, and matrices whose minors of a given size have the same sign are called *sign regular*. Sign regular matrices of signature $\{1, -1, 1, -1, \ldots\}$ are have positive entries, negative 2×2 minors, and so on. They can be described as "totally nonnegative" with columns in reverse order.

New results

Computing the eigenvalues of a matrix in floating point arithmetic using conventional algorithms (e.g., LAPACK) can cause the smallest eigenvalues and the Jordan structure to be lost to roundoff. In contrast, our new algorithm computes all eigenvalues of the above sign regular

matrices to high relative accuracy in floating point arithmetic. The zero eigenvalues and zero Jordan blocks are computed *exactly*.

The sign regular matrix be represented as a product of bidiagonals and the reverse identity. This merely means that its structure must be explicitly revealed. Formulas for the bidiagonal decompositions of the classical sign regular matrices (e.g., column-reversed Vandermonde, Cauchy, etc.) are readily available.

The algorithm reduces the sign regular matrix to a symmetric anti-bidiagonal matrix with the same eigenvalues. The eigenvalues of the later are computed using the result of Demmel and Kahan [3] (and attaching the signs, which are known from theory).

Our algorithm is subtraction-free, which guarantees the high relative accuracy. The only way that a zero eigenvalue (as well as any zero quantity) is computed to high relative accuracy is if it is computed exactly as evidenced by the inequality

$$|\lambda - \hat{\lambda}| \leq O(\varepsilon)|\lambda_i|.$$

The sizes of the zero Jordan blocks are deduced from the ranks of the powers of the original matrix, which are computed exactly.

Acknowledgement

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Totally positive matrices in the theory of robust spectral clustering

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Keywords: totally positive matrices, diagonal stability, Lyapunov equation, D-positivity, spectral clustering

The problem of establishing $\mathfrak{D}$-stability, i.e. spectra localization inside a prescribed subregion $\mathfrak{D} \subset \mathbb{C}$ is of major importance in the theory of control. In most cases, we are interested in *robust $\mathfrak{D}$-stability*, i.e. when the property of $\mathfrak{D}$-stability is preserved under certain perturbations of the initial matrix (polynomial).

In this talk, we consider totally positive matrices from the following different points of view:

- as a class of matrices, whose spectral properties are preserved with respect to some prescribed types of perturbations;

- as a class of matrices, which describes the type of uncertainty.

We consider spectra localization of totally positive matrices with respect to different stability regions. We study totally positive matrices as well as their subclasses (e.g. Jacobi and Vandermonde matrices), whose spectra localization is preserved under perturbations of different types. In addition, we consider relations between totally positive matrices, Kotelyansky matrices, D- and diagonally stable matrices. We study generalizations of totally positive matrices (e.g. J-sign-symmetric matrices), that have real spectra. Based on the results of [1], D-positivity and D-reality of matrix spectra are studied.

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Totally positive h-Bernstein bases, bidiagonal decomposition, and applications

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Keywords: totally positive matrix, h-Bernstein basis, bidiagonal decomposition

Abstract

One of the properties of h-Bernstein bases (a generalization of the Bernstein basis for the space of polynomials of degree less than or equal to n), analyzed in [3], is the *variation diminishing property*, an important issue in the field of CAGD. This property is related to the fact that h-Bernstein bases are totally positive, which means that the corresponding collocation matrices (the *h-Bernstein-Vandermonde matrices*) are totally positive.

The aim of this talk is to present an algorithm for the accurate computation of the bidiagonal decomposition of h-Bernstein-Vandermonde matrices, and then to use that bidiagonal decomposition as the starting point for the solution of several linear algebra problems with these matrices, by using some of the algorithms presented in [1]. The particular case where $h = 0$ corresponds to the Bernstein basis, a case which has been studied in detail in [2].

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On accurate computations with subclasses of totally positive matrices

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Keywords: totally positive matrices, high relative accuracy, bidiagonal factorization

Introduction

The bidiagonal decomposition of a nonsingular totally positive matrix provides a natural parametrization to perform algebraic operations to high relative accuracy. In fact, if the bidiagonal decomposition is known with high relative accuracy, then the computation of the inverse of the matrix, of its eigenvalues or of its singular values can be also performed with high relative accuracy. However, the obtention of the bidiagonal decomposition to high relative accuracy has been got, up to now, only for a few subclasses of nonsingular totally positive matrices. Recent advances on this subject are presented in this talk. In particular, we show new subclasses of nonsingular totally positive matrices, relevant in several fields, for which the bidiagonal decomposition has been obtained with high relative accuracy.

On the number of zeroes and poles of functions generating Pólya frequency sequences

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Keywords: totally positive sequences, Hurwitz matrices, Toeplitz matrices.

Abstract

A sequences of real numbers $\{a_n\}_{n=0}^{\infty}$ is called a *totally positive sequence*, or a *Pólya frequency sequence* if the Toeplitz matrix

$$T = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \dots \\ 0 & a_0 & a_1 & a_2 & \dots \\ 0 & 0 & a_0 & a_1 & \dots \\ 0 & 0 & 0 & a_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is *totally nonnegative* (that is all minors of the matrix are nonnegative). In this and only this case [1]-[2], the series $\sum\limits_{n=0}^{\infty} a_n z^n$ converges to a function of the following form:

$$R(z) = Cz^j e^{\eta z} \cdot \frac{\prod\limits_{\mu=1}^{M_1} \left(1+\frac{z}{\alpha_\mu}\right)^{k_\mu}}{\prod\limits_{\nu=1}^{N_1} \left(1-\frac{z}{\beta_\nu}\right)^{l_\nu}}, \tag{1}$$

where $C > 0$, $j \in \mathbb{N}$, $\eta, \alpha_\mu, \beta_\nu$ are positive and k_μ, l_ν are positive integers for all μ, ν. In (1), the numbers of distinct negative zeros M_1 and poles M_2 can be finite or infinite, which we denote using the inequality $0 \leqslant M_1, M_2 \leqslant \infty$.

However, the matrix T cannot help to specify the exact number of poles or/and zeroes of the function $R(z)$. In the talk, we present another matrix formed with the coefficients of the series $\sum_{n=0}^{\infty} a_n z^n$ which allows to determine whether the function $R(z)$ (the sum of the series) is an entire or meromorphic function.

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SPECIAL SESSION

Tropical matrix algebra and its applications

Organisers: Aljoša Peperko[1], Sergeĭ Sergeev[2]

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Tropical matrix algebra is a vibrant new area in mathematics, which has been developing since 1960's. The motivations of tropical matrix algebra are both applied (in particular, theory of optimal scheduling and discrete event systems) and pure, as there is a correspondence principle (Litvinov and Maslov) saying that every useful result and construction of traditional mathematics over fields might have a useful tropical counterpart. Therefore, tropical mathematics events traditionally bring together mathematicians of various backgrounds, both pure and applied. The emphasis of this workshop will be on new useful constructions in tropical matrix algebra, and possibly also on the influence of tropical geometry.

Linear algebra and convexity over symmetrized semirings, hyperfields and systems

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Keywords: tropical algebra, max-plus algebra, supertropical algebra, symmetrization, hyperfields, systems, convexity, linear algebra

Introduction

Rowen introduced in [4] a notion of algebraic structure, called systems, which unifies symmetrized tropical semirings [1], supertropical semirings [3], and hyperfields [5]. We study here linear algebra and convexity over systems [2].

Basic properties

Several notions of convexity are defined over systems, depending on the way equalities are weakened, for instance by replacing them by a balance equation or a surpassing relation. They can sometimes be related to the image of convex sets by generalized notions of valuations over the field of Puiseux series.

Main results

We discuss the relation between matrix ranks and convexity notions, like the Helly and Carathéodory numbers. We also compute upper and lower bounds for Helly numbers over various systems.

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Complete solution of tropical vector inequalities using matrix sparsification

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Keywords: tropical semifield, linear inequality, matrix sparsification, complete solution, backtracking

We consider linear vector inequalities defined in the framework of a linearly ordered tropical semifield (a semiring with idempotent addition and invertible multiplication). The problem is to solve two-sided inequalities, which have an unknown vector included in both sides, each taking the form of a given matrix multiplied by this unknown vector. Observing that the set of solutions is closed under vector addition and scalar multiplication, we reduce the problem to finding a matrix whose columns generate the entire solution set.

We represent the solution as a family of subsets, each defined by a matrix that is obtained from the given matrices by using a matrix sparsification technique. The technique exploits sparsified matrices to derive a series of new inequalities, which admit a direct solution in the form of matrices that generate their solutions. We describe a backtracking procedure that reduces the brute-force search of sparsified matrices by skipping those, which cannot provide solutions, and thus offers an economical way to obtain all subsets in the family. The columns in the generating matrices for subsets are combined together to form a matrix, which is further reduced to have only columns that constitute a minimal generating system of the solution. We use the reduced matrix to represent a complete exact solution of the two-sided inequality under consideration in a compact vector form.

We illustrate the results with numerical examples. Extension of the approach to solve two-sided equations is also discussed.

On Barabanov norms and subeigencones for max algebraic inclusions, and opacity for systems over the max algebra

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Keywords: joint spectral radius, extremal norm, Barabanov norm, opacity.

Introduction and overview

In the study of the joint spectral radius (JSR) in conventional algebra, the concepts of extremal and Barabanov norms play an important role. A classical result of Barabanov states that if the set $\Psi \subseteq \mathbb{R}^{n\times n}$ is compact and has no non-trivial invariant subspace (representation-theoretic irreducible), then it admits a Barabanov norm.

In this talk, we shall first discuss results on the existence of an extremal norm for a compact set Ψ of nonnegative matrices under a combinatorial requirement of irreducibility, before describing corresponding results for inclusions defined over the max algebra. In particular, we shall show that under an appropriate definition of irreducibility, a Barabanov norm always exists for such inclusions and provide an explicit characterisation of such a norm.

A set Ψ has the finiteness property if there exists some product P in the associated semigroup $\Sigma(\Psi)$ with $\rho(P) = \rho(\Psi)^k$ where k is the length of the product P, $\rho(P)$ its spectral radius, and $\rho(\Psi)$ the JSR. It has been known for some time that the finiteness property does not hold for a general finite set of matrices over the conventional algebra.

In contrast, it is not hard to show that it does hold over the max algebra. A spectrum maximising product is a matrix P in $\Sigma(\Psi)$ with $\rho(P) = \rho(\Psi)^k$. We will describe a number of results on the relationship between the eigencones and subeigencones of such products P and those of the matrix $S = \bigoplus_{A\in\Psi} A$. We will also briefly discuss a max algebraic analogue of a question posed by I. D. Morris concerning the uniqueness of Barabanov norms.

Recently, the concept of *opacity* has been extended from the setting of discrete event systems (DES) to continuous-state linear systems in [3]. For DES, results characterising the relationship between detectability, observability and opacity have been derived. In the final part of the talk, we will discuss appropriate definitions of opacity, detectability and observability for max algebraic systems and present some preliminary results clarifying the relationship between these concepts.

Acknowledgement

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Tropical matrices: ranks, powers and semigroup identities

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Keywords: tropical matrices, tropical rank, factor rank, semigroup identities, varieties of semigroups

Abstract

Tropical matrices are matrices over the semiring $\mathbb{T} := (\mathbb{R} \cup \{-\infty\}, \max, +)$. The set of such $n \times n$ matrices form the multiplicative monoid $\mathcal{M}_n(\mathbb{T})$.

Semigroup identities are linked to growth of semigroups; although the links are more intricate than in the case of groups. Existence of a semigroup identity for matrices can be understood as a pair of words indistinguishable by any weighted automaton.

Semigroup identities have been found for some submonoids of $\mathcal{M}_n(\mathbb{T})$, including triangular matrices, as well as for $\mathcal{M}_2(\mathbb{T})$ and $\mathcal{M}_3(\mathbb{T})$. In [3], we have proved the existence of identities for all $n \times n$ tropical matrices, for any n, to wit:

Theorem 1. *The monoid $\mathcal{M}_n(\mathbb{T})$ satisfies a nontrivial semigroup identity for every $n \in \mathbb{N}$. The length of this identity grows with n as $e^{Cn^2+o(n^2)}$ for some $C \leq 1/2 + \ln(2)$.*

The proof, presented in [3], is inductive and deeply relies on the relationships between the non-coincide notions of ranks for tropical matrices (see [1] for details). It combines three main ingredients:

1. A lemma from [5] which reduces the problem to matrices having maximal factor rank;
2. An adapted theorem from [2] which deals with matrices having maximal tropical rank, using identities of triangular matrices;
3. New relationships between the factor rank of a large enough power of a matrix and the tropical rank of the original matrix.

The proof of the latter is based on the so-called weak CSR expansion – a method developed by T. Nowak, S. Sergeev, and the second author in [4]. The main results in this part are the following.

Theorem 2. *Any* $A \in \mathcal{M}_n(\mathbb{T})$ *satisfies* $\mathrm{rk}_{\mathrm{fc}}(A^t) \leq \mathrm{rk}_{\mathrm{tr}}(A)$ *for any* $t \geq (n-1)^2 + 1$.

Proposition 3. *If* $A \in \mathcal{M}_n(\mathbb{T})$ *satisfies* $\mathrm{rk}_{\mathrm{tr}}(A^{\overline{n}}) < n$, *with* $\overline{n} = \mathrm{lcm}\,(1, \cdots, n)$, *then* $\mathrm{rk}_{\mathrm{fc}}(A^{t\overline{n}}) < n$ *for any* $t \geq 3n - 2$.

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Tropical planar networks

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Keywords: planar networks; total positivity; total nonnegativity; compound matrix; permanent.

Introduction

The combinatorial properties of minors of the weight matrix associated with a planar network are well known [2]. These were studied by Karlin and McGregor back in the 50's (see [6]). Some applications were given by Gessel and Viennot in [4], [5]. In this context, totally nonnegative matrices arise as weight matrices of planar networks. We show that, as consequence of earlier results, the same is true in the tropical setting.

Basic properties

This work is a direct extension of [3], which discussed the tropical analogues of totally positive and totally nonnegative matrices, inspired by [1]. These arise when considering the images by the nonarchimedean valuation of the corresponding classes of matrices over a real nonarchimedean valued field, like the field of real Puiseux series. It has been shown that the nonarchimedean valuation sends the totally positive matrices precisely to the Monge matrices. That led to explicit polyhedral representations of the tropical analogues of totally positive and totally nonnegative matrices. Also, tropical totally nonnegative matrices with a finite permanent shown to be factorized in terms of elementary matrices. The eigenvalues of tropical totally nonnegative

matrices were determined and related with the eigenvalues of totally nonnegative matrices over nonarchimedean fields.

Main results

Inspired by [2], we study the relation between planar networks and total positivity. More precisely, we show that every totally connected planar network with real weights has a tropical totally nonnegative weight matrix. In particular, we provide an explicit condition for the weight matrix to be tropical totally positive. Moreover, for every square tropical totally nonnegative matrix A there exists a planar network such that A is its weight matrix. In particular, if A is tropical totally positive, then the planar network is unique.

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The Bonsall and lower cone spectral radius of suprema preserving mappings and approximate point spectrum

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Keywords: cone mappings, approximate point spectrum, max algebra

Introduction

We will present results on Bonsall's and lower cone spectral radius of positively homogeneous, bounded or Lipschitz and suprema preserving mapping on max cones in normed vector lattices and its approximate point spectrum.

Our results apply to max type kernel operators. In the special case of (infinite) matrices over max algebra we obtain additional results.

Basic notations

Let $T\colon C \to C$ be a positively homogeneous, Lipschitz mapping that preserves finite suprema, where $C \subset X_+$ is max-cone in (a possibly infinite dimensional) normed vector lattice X (a normed space, which suitably respects the order).

Let $r(T)$, $d(T)$, $r_x(T)$, $\sigma_{ap}(T)$ denote the Bonsall cone spectral radius, lower cone spectral radius, local spectral radius at $x \in C$ and approximate point spectral radius, respectively.

Main results

In particular, we show that under suitable assumptions $r(T)$, $d(T)$, $r_x(T)$ for $x \neq 0$ are included in $\sigma_{ap}(T)$. We also show that the max polynomial spectral mapping theorem holds for $\sigma_{ap}(T)$.

We apply our results to max type integral operators and show additional results in the special case of of (infinite) matrices over max algebra.

Acknowledgement

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Some geometry of the SMPA

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Keywords: symmetrized max-plus algebra, metric, Chebyshev set, convex set

This talk is based on [1]. We assume only the axiomatic system of Zermelo-Fraenkel with no use of the Axiom of Choice.

The symmetrized max-plus algebra $\mathbb{S}$ is considered with an embedding, called the *canonical embedding*, into $\mathbb{C}$ with two basic metrics: the *Euclidean metric* d_e and the *inner metric* d_{in}, both inducing the same topology (other realizations of $\mathbb{S}$ as subsets of $\mathbb{C}$ are possible).

It appears that $\otimes$ is continuous and $\oplus$ is only almost everywhere continuous as a mapping from $\mathbb{S} \times \mathbb{S}$ to $\mathbb{S}$.

Question 1. *What is the most natural topological structure on* $\mathbb{S}$*?*

We produce the *Euclidean metric* D_e and the *inner metric* D_{in} on $\mathbb{S}^n$ and consider *traditional segments*, *geometric segments* and *semimodule segments* between points as well as respective convexities ($\mathbb{S}^n$ is seen as a semimodule over $\mathbb{R}_{\max}$). A geometric segment in $\mathbb{S}$ is the intersection of all connected subsets of $\mathbb{S}$ that contain both ends of the segment. A geometric segment in $\mathbb{S}^n$ is decomposable into no more than $n+1$ traditional segments. Traditionally convex and geometrically convex sets in $\mathbb{S}^n$ are connected.

On the other hand, semimodule segments in $\mathbb{S}$ can have up to 3 connected components and need not be closed. A semimodule segment in $\mathbb{S}^2$ can have 5 connected components.

Semimodule segments in $\mathbb{S}$ are not necessarily Chebyshev sets. Nonempty closed semimodule convex sets in $\mathbb{S}$ can admit up to 3 nearest

points to a given point (relative to d_e or d_{in}), and products of such sets in $\mathbb{S}^n$ up to 3^n nearest points to a given point (relative to D_e or D_{in}).

Theorem 1. *For a set $C \subseteq \mathbb{S}$, the following conditions are equivalent:*

a) C is connected,

b) C is geometrically convex,

c) C is d_e-Chebyshev,

d) C is d_{in}-Chebyshev.

Question 2. *Is geometric convexity of a closed non-empty subset in $\mathbb{S}^n$ ($n > 1$) equivalent to being D_e-Chebyshev or D_{in}-Chebyshev?*

For a product of n closed non-empty subsets of $\mathbb{S}$, being D_e-Chebyshev or D_{in}-Chebyshev is equivalent to being connected.

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Quasi–Euclidean classification of maximal alcoved convex polyhedra

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Keywords: tropical algebra, max–plus algebra, normal matrix, idempotent matrix, visualized matrix, alcoved polyhedron, dodecahedron, convex body, quasi–Euclidean class, perturbation, invariant, edge–length, tropical distance

Introduction

Tropical geometry blends metric geometry with combinatorics. The topic of this talk is the family of alcoved polyhedra: a rich, very well–behaved and beautiful class of convex 3–dimensional bodies. They have facet equations only of two types: $x_i = cnst$ and $x_i - x_j = cnst$. Three consequences follow. First, the f–vector and the the facet sequence are restricted (combinatorial restrictions). Second, the facet angles and the dihedral angles are restricted (Euclidean restrictions). Third, the constant terms of the facet equations can be assembled into a matrix (algebraization via tropical–multiplicatively idempotent normal matrices).

In this talk, we classify alcoved polyhedra from a topological, affine and Euclidean point of view, i.e., we transform them by small perturbations, affine maps and space symmetries. Here is what we do. First, we declare all boxes to be equivalent (disregarding length). Having done so, we are left with the classification of alcoved polyhedra having a common bounding box. Say this common bounding box is $\mathcal{Q}$, the unit cube centered at the origin. Two alcoved polyhedra $\mathcal{P}$ and $\mathcal{P}'$ whose

bounding box is $\mathcal{Q}$ are equivalent for us, in two cases: if there exists a symmetry of $\mathcal{Q}$ taking $\mathcal{P}$ bijectively onto $\mathcal{P}'$ or if $\mathcal{P}'$ is a sufficiently small perturbation of $\mathcal{P}$ (or a combination of both). Finally, when we want to compare two arbitrary alcoved polyhedra $\mathcal{P}$ and $\mathcal{P}'$, first we transform their bounding boxes $\mathcal{B}$ and $\mathcal{B}'$ into $\mathcal{Q}$ by bijective affine maps f and f'. If $f(\mathcal{P})$ and $f'(\mathcal{P}')$ are equivalent, as described above, then we define the original $\mathcal{P}$ and $\mathcal{P}'$ to be equivalent. It is an angle–preserving equivalence relation between maximal alcoved polyhedra. For this new notion, we have coined the expression *quasi–Euclidean equivalence*.

Maximality is considered wrt the f–vector, i.e., maximal alcoved polyhedra are dodecahedra having 20 vertices and 30 edges.

Main results

We prove the following theorem:

The quasi–Euclidean classification in the family of maximal alcoved dodecahedra in $\mathbb{R}^3$ has eight classes.

The proof is reached after the following steps. In each alcoved polyhedron $\mathcal{P}$, two distinguished vertices are called *North and South Poles*, and marked $\mathcal{N}$ and $\mathcal{S}$. After an idea of Kepler's, *the polyhedron $\mathcal{P}$ is the union of three parts: North Cask, South Cask and Equatorial Belt.* Each Polar Cask has a *Cask type*. Cask types are described by a *vector* and a *chirality word*. In the maximal case, the vector is $(p.q.r)$, with $p, q, r \in \{4, 5, 6\}$ and $p + q + r = 15$. The chirality word is either *left* or *right*. In the maximal case, we show that both the Equatorial Belt and the quasi–Euclidean class are determined only by the North and South Cask types. The Cask types are determined only by the signs of six 2–minors of the normal idempotent matrix A representing $\mathcal{P}$ i.e., $\mathcal{P} = \mathcal{P}(A)$. Ultimately, the Cask types are determined by the signs of a 6–tuple, called *difference tuple*, which is computed from the *perturbation matrix* E of A. The matrix $B = A + E$ provides the bounding box, i.e., the bounding box of $\mathcal{P}(A)$ is $\mathcal{P}(B)$.

Acknowledgement

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Polynomial convolutions in max-plus algebra

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Keywords: max-plus algebra, max-convolution of maxpolynomials, Hadamard product, characteristic maxpolynomial

Introduction

Recently, Marcus, Spielman and Srivastava [2], [1], following their solution of the Kadison-Singer problem [3], came to study certain convolutions of polynomials. They established a strong link to free probability by showing that these convolutions capture the expected characteristic polynomials of random matrices. In addition, these convolutions preserve the property of the roots being real numbers.

We explore analogues of these types of convolution polynomials in the setting of max-plus algebra, where the max-permanent replaces the determinant and the maximum is the analogue of the expected value.

Basic properties

The basic max-plus structures and notions we use are (in brackets their analogues in the standard setting): max convolution (additive convolution), Hadamard product (multiplicative convolution), maximum (expectation), permutation max-plus matrices (orthogonal matrices), principally-dominant matrices (diagonizable matrices), full characteristic maxpolynomial (characteristic polynomial).

Main results

We obtain formulas that are similar to those of [2] for the convolution of characteristic maxpolynomials. However, whereas in the standard setting only bounds on the maximal roots are known, here we get a simple description of all the roots of the convolution maxpolynomial in terms of the roots of the involved maxpolynomials p and q of degree n: the maximal n roots among those of p and q (in max convolution) and the product (sum in standard arithmetic) of the ordered lists of the roots of p and q (in the Hadamard (multiplicative) convolution). The preservation of the real-rootedness is translated here to maxpolynomials in full canonical form (FCF) and we show that the full characteristic maxpolynomial is in FCF.

Acknowledgement

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Geometry and linear algebra on systems

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Keywords: negation map, system, triple, tropical, linear algebra, geometry

Introduction

As part of an ongoing project to present an axiomatic algebraic theory which unifies and "explains" aspects of tropical algebra, hyperfields, and fuzzy rings by embedding T in a larger set $\mathcal{A}$ endowed with more structure, we consider the affine geometric aspects and relate them to Lorscheid's semigroup semialgebra. We also have results on linear algebra, done jointly with Akian and Gaubert.

Basic properties

The keys to this theory are a general "negation map," as well as a "surpassing relation" which generalizes equality; together with $\mathcal{A}$ and T these are called a "system," which provides a framework for analogs of theorems from classical algebra.

The set of prime congruences on $\mathcal{A}$ provides a Zariski topology, which can be used as the foundation for affine geometry.

Main results

Our main results concern basic invariants of geometry, as well as conditions guaranteeing the equality of matrix rank and row rank (as well as counterexamples).

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Cryptography using tropical matrix algebra

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Keywords: cryptography, Stickel protocol, tropical, commuting matrices

Introduction

In cryptography, Stickel's protocol is used for generating a secret key shared between Alice and Bob. Grigoriev and Shpilrain [1] suggested a tropical implementation of this protocol by means of tropical matrix polynomials. In response, Kotov and Ushakov [3] suggested an attack on this new implementation. Their attack works well, however only for small maximal degrees of tropical matrix polynomials. Our idea is to develop new protocols that will be similar to Stickel's protocol, by means of the recently found classes of tropical commuting matrices.

Main results

Jones [2] considered square matrices over $\mathbb{R}_{\max}$ that satisfy the relation $a_{ij} \otimes a_{jk} \leq a_{ik} \otimes a_{jj}$. If one defines $A^{(\alpha)} = (a_{ij}^{(\alpha)})_{i,j=1}^{n}$ by

$$a_{ij}^{(\alpha)} = a_{ij} \otimes (a_{ii} \oplus a_{jj})^{\alpha-1},$$

then such matrices commute for any $\alpha\colon 0 \leq \alpha \leq 1$. This allows us to use $A^{(\alpha)}$ instead of matrix powers in the implementation of Grigoriev and Shpilrain. We then modify the Kotov-Ushakov attack for application to this case, and we analyze how it performs.

Developing an observation of Linde and de la Puente [4], we denote by $[2r, r]_n^k$ the set of matrices A such that $a_{ii} = k$ for all i and $k \geq 0$ and $a_{ij} \in [2r, r]$ for $i \neq j$. We show that A and B commute for $A \in [2r, r]_n^{k_1}$ and $B \in [2s, s]_n^{k_2}$, where $r, s \geq 0$ and $a_{ii} = k_1 \geq 0$, $b_{ii} = k_2 \geq 0$. Using this class of commuting matrices, we suggest the following protocol:

1. Alice and Bob agree on a public matrix $W \in \mathbb{R}_{\max}^{n \times n}$;

2. Alice chooses matrices $A_1 \in [2a, a]_n^{k_1}$ and $A_2 \in [2b, b]_n^{k_2}$, and then she sends $U = A_1 \otimes W \otimes A_2$ to Bob.
   Bob chooses matrices $B_1 \in [2c, c]_n^{l_1}$ and $B_2 \in [2d, d]_n^{l_2}$, and then he sends $V = B_1 \otimes W \otimes B_2$ to Alice.

3. Alice computes her secret key $K_{Alice} = A_1 \otimes V \otimes A_2$. Bob computes his secret key $K_{Bob} = B_1 \otimes U \otimes B_2$. We have $K_{Alice} = K_{Bob}$.

We show that the Kotov-Ushakov attack can be generalized to deal with this protocol. We also describe some easier attacks that work in important special cases, and discuss some other protocols that use tropical matrix algebra.

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Two-sided equation (max,+)/(min,+)-systems

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Introduction

Algebraic structures used in this contribution are so called $(\max,+)$- and $(\min,+)$-algebras, which have appeared in the literature approximately since the 60-ties of the last century (see [1]–[5] and the references there in). In these structures the usual addition and multiplication used in the classic linear algebra are replaced either by operations $(\max,+)$ or $(\min,+)$. The semigroup extremal operation max or min replaces addition and opration + replaces the multiplication. The operations are considered on the set of real numbers R and extended to elements of R^n. Multiplication of matrices and vectors is extended by analogy with the usual linear algebra taking into acount the replacement of the operations. In this way we can consider systems of $(\max,+)$- or $(\min,+)$-linear equations and/or inequalities. Systems with variables only on one side of the relations are called one-sided systems, systems, in which variable occur on bothe sides are called two-sided. The necessity to make a difference between the two types of relations follows from the fact that the extremal operations replacing the addition are semigroup operations, which exclude the possibility to transfer variables from one side of the relation to the other. The treatment of one-sided and two-sided relations therefore proved to be substantially different.

Notations, problem formulation

We introduce the following notations: $I = \{1, \ldots, m\},\ J = \{1, \ldots, n\}$, $x^T = (x_1, \ldots, x_n),\ y^T = (y_1, \ldots, y_n)$, $A\ o\ x = ((A\ o\ x)_1, \ldots, (A\ o\ x)_m)$, $B\ o'\ y = ((B\ o'\ y)_1, \ldots, (B\ o'\ y)_m)$, where

$$(A\ o\ x)_i = \max_{j \in J}(a_{ij} + x_j), \quad (B\ o'\ y)_i = \min_{k \in J}(b_{ik} + y_k),\ i \in I$$

We will investigate solvability of the equation system with respect to $x, y \in R^n$

$$A\ o\ x\ =\ B\ o'\ y,$$

i.e. componentwise

$$\max_{j \in J}(a_{ij} + x_j) = \min_{k \in J}(b_{ik} + y_k),\ i \in I.$$

Besides the case of inequalities $A\ o\ x \leq B\ o'\ y$, as well as systems with $x = y$ and its relations to so called steady state solutions will be briefly discussed.

Main results

Let $Q = (q_{jk}) = -A^T\ o'\ B$, i.e. $q_{jk} = \min_{i \in I}(b_{ik} - a_{ij}),\ \forall j, k \in J$. The following implication can be proved:

$$A\ o\ x \leq B\ o'\ y \quad \Rightarrow \quad x \leq \overline{x}(y) \equiv Q\ o'\ y.$$

Conditions under which $(\overline{x}, y)$ is a solution of equation system $A\ o\ x = B\ o'\ y$ will be derived. Relations of the results to modified systems $A\ o\ x = B\ o'\ x$, $A\ o\ x = B\ o\ y$ will be studied. Possible application of the results to machine-time and departure-arrival scheduling will be briefly discussed.

Acknowledgements

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SPECIAL SESSION

Recent developments of verified numerical computations

Organisers: Takeshi Ogita[1], Siegfried M. Rump[2]

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[2] Hamburg University of Technology

This special session is devoted to verified numerical computations, in particular, verification methods for linear algebra, optimization, and even for ordinary differential equations. Since verified numerical computations enable us to rigorously solve mathematical problems by numerical methods in pure floating-point arithmetic, they became increasingly important in a wide range of science and engineering. The main objective of the special session is to discuss several recent topics on verification methods and related numerical analysis and matrix methods.

Verified inclusion of a nearby matrix of specified rank deficiency

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Keywords: verified error bounds, distance to rank deficiency, ill-posedness, INTLAB

Introduction

Given a matrix $A \in \mathbb{R}^{m \times n}$ with $m \geq n$ and an integer k, we are looking for a method to compute rigorous bounds for a perturbation of A such that the perturbed matrix has at least rank deficiency k.

A natural approach uses the singular value decomposition $U\Sigma V^T$ of A. Let Σ_k denote the matrix obtained from Σ by setting the k smallest diagonal entries to zero. Then $B := U\Sigma_k V^T$ is a matrix with at least rank deficiency k approximating A. Indeed, if the distance is measured via a unitarily invariant norm, this approximation is best possible [1].

Rigorous bounds for approximate solutions

By computing an approximate singular value decomposition $A \approxeq \widetilde{U}\widetilde{\Sigma}\widetilde{V}^T$ and calculating $\mathbf{B} \supseteq \widetilde{U}\widetilde{\Sigma}_k\widetilde{V}^T$ under usage of an interval floating-point arithmetic, one derives a verified inclusion for a matrix nearby A with at least rank deficiency k.

A major issue of this approach is that the evaluation of $\widetilde{U}\widetilde{\Sigma}_k\widetilde{V}^T$ introduces intervals with comparably large absolute diameters. We will show how to compute tighter inclusions using the following result.

Lemma 1. *Let $W \in \mathbb{R}^{n\times k}$ be given, abbreviate $G := I - W^*W$ and assume $\|G\|_2 \le \alpha < 1$. Then there exists $\Delta \in \mathbb{R}^{m\times n}$ satisfying*

$$\Delta = AWW^* + F_1 = AW(I+G)W^* + F_2 \text{ with } \|F_\nu\|_2 \le \frac{\alpha^\nu}{\sqrt{1-\alpha}}\,\|AW\|_2$$

such that $A - \Delta$ has at least rank deficiency k.

Verified bounds for optimal solutions

Verification methods use standard floating-point arithmetic to estimate possible rounding errors rigorously. They are usually fast, however, due to the use of floating-point arithmetic, their application is basically restricted to well-posed problems [2].

Computing a tight verified inclusion of the actual best approximation $B = U\Sigma_k V^T$ is much more difficult than the above problem; it actually becomes ill-posed if the $(n-k)$-th singular value has a higher multiplicity. The reason for the latter is that the space of corresponding singular vectors is higher dimensional. For arbitrarily small $\varepsilon > 0$ there exists an ε-perturbation of the matrix with mutually different singular vectors. These problem instances lie outside of the scope of verification methods.

We present methods for the well-posed cases and discuss the surprising fact that the problem to compute a tight inclusion for a perturbation with minimal 2-norm is always well-posed.

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A note on verification methods for sparse non-symmetric linear systems

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Keywords: sparse non-symmetric linear system, error bounds

Introduction

In this presentation, we consider the accuracy of a computed solution $\tilde{x}$ of a sparse linear system

$$Ax = b, \tag{1}$$

where A is a real $n \times n$ matrix and b is a real n-vector. Sparse linear systems arise in scientific and engineering problems. To preserve sparsity is important from the view of memory and computation efficiency in this problem. Rump and Ogita [1] proposed a method to verify a lower bound of the minimum eigenvalue of A in the case where A is symmetric and positive definite and gave rigorous error bounds of a computed solution of a linear system. The method requires only a Cholesky factorization of $A - sI$ where s is a positive constant. If the Cholesky factor of the matrix is sparse, then the method is fast. Moreover, the method can be applied to the normal equation

$$A^T Ax = A^T b \tag{2}$$

of (1). However, a condition number of $A^T A$ is the square of that of A, i.e., $\kappa(A^T A) = \kappa(A)^2$, and the Cholesky factorization of $A^T A$ often fails when $\kappa(A) > 10^8$. We consider (1) and compare verification methods with LU factorazation $PAQ = LU$.

We first consider a verification method based on the inequality

$$|A^{-1}b - \tilde{x}| \le |(RA)^{-1}| \, |R(b - A\tilde{x})|, \tag{3}$$

where R is an approximate inverse of A. If R and RA are sparse, the verification method based on (3) is fast. Second, we consider a verification method based on the inequality

$$|A^{-1}b - \tilde{x}| \le |\big(X_U\big(X_L\big(PAQ\big)\big)\big)^{-1}| \, |X_U\big(X_L\big(b - A\tilde{x}\big)\big)|, \tag{4}$$

where X_L and X_U are approximate inverses of L and U, respectively. If X_U, X_L, and $X_U\big(X_L\big(PAQ\big)\big)$ are sparse, the verification method based on (4) is fast. Finally, we consider a verification method based on the inequality

$$|A^{-1}b - \tilde{x}| \le |X_U||\big(\big(X_L\big(PAQ\big)\big)X_U\big)^{-1}| \, |X_L(b - A\tilde{x})|. \tag{5}$$

We will show numerical results with sparse matrices in the SuiteSparse Matrix Collection [2], verification methods based on (3) or (4) cannot preserve sparsity in many cases, and the verification method based on (5) preserves sparsity in some examples. We will show that a large problem ($n \ge 100,000$) can be verified by the verification method based on (5) and this large problem cannot be transformed into a block upper triangular matrix which has only small blocks by Dulmage–Mendelsohn decomposition.

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Accurate and verified solutions of large sparse linear systems arising from 3D Poisson equation

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Keywords: verified numerical computations, sparse linear systems

Introduction

We study verified solutions of linear systems arising from the following type of 3D Poisson equation

$$-\nabla \cdot (\lambda \nabla u) = f$$

with Dirichlet boundary conditions. A typical example is the steady-state heat equation where u is the temperature, λ is the thermal conductivity, and f is the heat-flux density of the source.

In numerical computations, we often discretize the equation by the finite difference method or the finite element method. Then we obtain a sparse linear system $Ax = b$, where the coefficient matrix A is expected to be monotone, i.e., all the elements of A^{-1} are nonnegative, from the physical condition of the problem. To solve such linear systems, iterative solution methods such as the conjugate gradient (CG) method and its variants are frequently used. In such cases, we usually measure a residual norm for checking the convergence. However, we do not know the accuracy of computed solutions. Methods of calculating error bounds of computed solutions are so-called verification methods. Excellent overview can be found in [3].

In [2], a fast verification method is proposed for calculating an error bound ε of a computed solution $\widehat{x}$ of a sparse linear system $Ax = b$ with A being monotone, satisfying

$$\|A^{-1}b - \widehat{x}\| \leq \varepsilon.$$

The method is independent of iterative solvers and can be applied to the case where A is sparse.

In this study, we adapt the verification method to high-performance computing (HPC) environments. For this purpose, we modify several points in terms of both the quality of the verified error bounds and the speed of the verification process: we tighten the computed error bounds using the approach proposed in [1] with high-precision residual computation and speed up the verification process by reducing the memory access.

Numerical results will be presented with some applications.

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Rigorous results in electronic structure calculations

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Keywords: optimization, electronic structure calculation, verification

Introduction

Electronic structure calculations, in particular the computation of the ground state energy E_0, lead to challenging problems in optimization. These problems are of enormous importance in quantum chemistry for the calculation of properties of N-electron molecules.

Basic properties

Minimization methods for computing E_0 can be developed by employing a variational approach, where γ and Γ, the first- and second-order reduced density matrix (RDM), define the variables. This concept leads to large-scale semidefinite programming (SDP) problems, that are constrained by N-representability conditions [1]. The resulting SDP problems provide a lower bound for E_0. Upper bounds of E_0 can be calculated for example with the Hartree-Fock method.

Main results

With the a priori knowledge of elementwise bounds on the variables γ and Γ it is possible to compute rigorous error bounds for the resulting SDP problems, which consider all rounding errors [2]. By the following lemma we derived in [1] such elementwise a priori bounds, that use the well-known maximal eigenvalues $\lambda_{\max}(\gamma) = 1$ and $\lambda_{\max}(\Gamma) = N$.

Lemma 1. *Let γ be the symmetric positive semidefinite 1-RDM, then*

$$|\gamma(i,i')| \leq \begin{cases} \frac{1}{2}\lambda_{\max}(\gamma), & \text{for } i \neq i', \\ \lambda_{\max}(\gamma), & \text{for } i = i'. \end{cases}$$

Lemma 1 can be applied to Γ as well. With lemma 2 we derived in [1] tighter bounds for $\widetilde{\Gamma}$, the compacted form of Γ, which is actually used in the SDP relaxations.

Lemma 2. *The eigenvalues of the compacted matrix $\widetilde{\Gamma}$ and Γ satisfy*

$$\lambda^{\downarrow}(\widetilde{\Gamma}) = \frac{1}{2}\lambda^{\downarrow}(\Gamma).$$

In our numerical results in [1] we successfully treated problem instances with one hundred thousand to 20 million variables and with thousand to thirty thousand constraints. It turned out that in almost all cases eight decimal digits of the optimal value were correct. The additional effort for computing the rigorous bounds was negligible compared with the effort for computing the approximations.

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Error-free transformation of a product of three matrices and its applications

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Keywords: floating-point arithmetic, matrix multiplication, eigenvalue problems

Introduction

This talk concerns an error-free transformation for a product of three matrices. Let $\mathbb{F}$ be a set of floating-point numbers defined in IEEE 754, see [1]. For three matrices $A \in \mathbb{F}^{m\times n}, B \in \mathbb{F}^{n\times p}$, and $C \in \mathbb{F}^{p\times q}$, we can obtain a matrix $B' \approx B$ such that no rounding error occurs in the evaluation of $AB'C$ by adopting error-free transformation of a matrix multiplication [3]. We apply this approach to test matrices with specified eigenvalues.

Test matrices for eigenvalue problems

Eigenvalue problems for a matrix $A \in \mathbb{R}^{n\times n}$ are important in linear algebra. Since there is no explicit method for solving a general problem for $n \geq 5$, numerical iterative methods are employed for this problem. If exact eigenvalues are known in advance, the information is useful in checking accuracy and stability of numerical algorithms for the eigenvalue problems, i.e., the exact relative error can be checked. A topic for test matrices is well summarized in Chapter 28 in [2]. Our goal is to generate test matrices with specified eigenvalues.

We develop two functions generating a matrix $A \in \mathbb{F}^{n\times n}$ based on XDX^{-1} using a matrix $D \in \mathbb{F}^{n\times n}$ and a non-singular matrix $X \in \mathbb{F}^{n\times n}$. We produce $D' \approx D$ such that $XD'X^{-1}$ can be computed without rounding errors. The feature of a function is as follows.

Input: a vector $d \in \mathbb{F}^n$ and an expected condition number c_1 of X.

Output: a matrix A, $X \in \mathbb{F}^{n\times n}$, two vectors p, $q \in \mathbb{F}^n$, and $c_2 \in \mathbb{F}$ for the condition number of X ($c_2 \approx c_1$).

Here, d_i is an expected eigenvalue of A, and the exact eigenvalues of A are $p_i + q_i \approx d_i$. The matrix X has the exact eigenvectors of A. The feature of another function is as follows.

Input: a matrix $D \in \mathbb{F}^n$.

Output: a matrix $A \in \mathbb{F}^{n\times n}$ and two vectors $p, q \in \mathbb{F}^n$.

The role of A, p and q is the same as the previous one. A candidate of the matrix D is an upper bi-diagonal matrix of the Jordan normal form. Another candidate of D is a block diagonal matrix with 2-by-2 blocks, that is used for generating a real matrix with complex eigenvalues. We will introduce the detail of the methods and numerical examples in the presentation.

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Gershgorin circles and the determinant of real or complex matrices

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Keywords: Gershgorin circle, determinant, Minkowski product

Abstract

Each connected component of the Gershgorin circles of a matrix contains exactly as many eigenvalues as circles are involved. Thus, the Minkowski (power set) product of all circles is an inclusion of the determinant if all circles are disjoint. First, we prove that statement to be true for real matrices even if their circles overlap.

For a real matrix, the midpoints of the Gershgorin circles are real and the determinant is real as well. Thus the statement can be reduced to prove that the Minkowski product of the projections of the Gershgorin circles on the real axis contains the determinant.

In case of complex matrices we have to investigate the Minkowski product of complex disks. This set is known to be bounded by the outer loop of a Cartesian oval. We first derive a parametrization of the outer loop of a Cartesian oval without case distinction. Based on that we prove that the determinant is included in the product of the Gershgorin circles also for a complex matrix.

Moreover, the product of complex disks is further investigated. Let D_R, D_r, D_S, D_s be complex disks with common center 1 and radii R, r, S, s. We derive surprisingly simple necessary and sufficient conditions for $A := D_R D_r$ being a subset or superset of $B := D_S D_s$. For

example, A is a subset of B if and only if the left-most real point of A is in B, and A is a superset of B if and only if the right-most real point of B is in A. Partially, this extends to n-fold disk products $D_1 \cdots D_n$ for $n > 2$.

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Sharp inclusions of the determinant of real or complex, point or interval matrices

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Keywords: Determinant; classical adjoint, verification methods, sharp bounds, extremely ill-conditioned matrices, NP-hard

Abstract. We discuss several methods to compute a verified inclusion of the determinant of a real or complex, point or interval matrix. For point matrices, large condition number 10^{15}, and large dimension ($n = 1000$) still highly accurate inclusions are computed. For real interval matrices we show that any vertex may be a unique extreme point. For wide radii we show that preconditioning may widen an inclusion significantly, and Hadamard's bound may be much better.

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Preconditioned Cholesky QR algorithms in an oblique inner product

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Keywords: QR decomposition, preconditioning methods

Introduction

Given a full column rank matrix $A \in \mathbb{R}^{m\times n}$ with $m \geq n$ and a symmetric positive definite matrix $B \in \mathbb{R}^{m\times m}$, we consider QR decomposition in an oblique inner product, which computes $Q \in \mathbb{R}^{m\times n}$ and $R \in \mathbb{R}^{n\times n}$ satisfying

$$A = QR, \quad Q^T BQ = I,$$

where R is an upper triangular matrix and I is the identity matrix. We introduce the CholeskyQR algorithm in an oblique inner product [1] using MATLAB-like notations.

function $[Q, R] = \mathrm{CholQR}(A, B)$
  $C = A' * B * A;$ % $C \approx A^T BA$
  $R = \mathrm{chol}(C);$ % $C \approx R^T R$
  $Q = A/R;$ % $Q \approx AR^{-1}$
end

If A or B is ill-conditioned, a floating-point Cholesky decomposition often breaks down. To solve this problem, we propose a preconditioning method for A. We will present numerical examples showing the efficiency and robustness of the proposed algorithms.

Preconditioned Cholesky QR algorithm

We propose a preconditioning method using Doolittle's LU decomposition for A such that $PA = LU$, where P is a permutation matrix, L is a unit lower triangular matrix, and U is an upper triangular matrix. The LU factors can efficiently be used for preconditioning [2].

function $[Q, R] = \mathrm{LU\_CholQR}(A, B)$
  $[L, U, p] = \mathrm{lu}(A, '\,\mathrm{vector}')$; % $A(p, :) \approx LU$
  $C = L' * B(p, p) * L$;
  $R = \mathrm{chol}(C) * U$;
  $Q = A/R$;
end

If A and B are dense matrices, $\mathrm{CholQR}(A, B)$ and $\mathrm{LU\_CholQR}(A, B)$ require $2m^2n + 3mn^2 + 2n^3/3$ and $2m^2n + 5mn^2$ floating-point operations, respectively. Therefore, if $m \gg n$, the proposed algorithms achieves high performance as much as the original Cholesky QR algorithm. Moreover, the proposed algorithm using double precision arithmetic is applicable to the case where A and B are ill-conditioned. Even if the condition numbers of A and B are nearly 10^{16}, the proposed algorithm can produce Q and R without breaking down.

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SPECIAL SESSION

Interval matrices

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Interval computation is a discipline addressed to handling and computing with interval data. The fundamental property of interval computation is the "enclosing property", guaranteeing that all possible realizations of interval data and all roundoff errors are taken into account. Due to this property, interval computation is an important technique for obtaining rigorous results, and for this reason it is used in numerical analysis, global optimization, constraint programming and many other areas. The key object in interval computation is an interval matrix, which is by definition a set of matrices lying entrywise between two given lower and upper bound matrices. This special session will be devoted to investigation of various properties of interval matrices, including theoretical characterization, developing efficient algorithms, classification in the computational complexity sense, and related problems such as solving interval linear systems of equations.

Pseudoinverses of interval matrices

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Keywords: interval analysis, interval matrix algebra, Moore-Penrose pseudoinverse, Greville algorithm

Introduction

One of the fundamental problems in interval linear algebra is to decide for some given interval matrix if it is regular. An interval matrix is *regular* if all its selections are regular.

In classical linear algebra, a natural approach in the case of singular matrix is to find a pseudoinverse matrix which is in some sense very close to being the inverse matrix. There is not a single universal way to introduce such notion and thus there are different kinds of pseudoinverses. One of them is the *Moore-Penrose pseudoinverse matrix*.

Definition 1. *For a given matrix $A \in \mathbb{R}^{m\times n}$, the Moore-Penrose generalized inverse is the matrix $A^+ \in \mathbb{R}^{n\times m}$ satisfying*

$$\begin{aligned} AA^+A &= A,\\ A^+AA^+ &= A^+,\\ (AA^+)^T &= AA^+,\\ (A^+A)^T &= A^+A. \end{aligned}$$

The above definition can be generalized to interval matrices in the following way.

Definition 2. *For any interval matrix* $\mathbf{A} \in \mathbb{IR}^{\mathbf{m}\times\mathbf{n}}$ *we define the* interval pseudoinverse matrix $\mathbf{A}^+ \in \mathbb{IR}^{m\times n}$ *as the minimal interval matrix so that* $\mathbf{A}^+ \supset \{A^+ : A \in \mathbf{A}\}$.

Talk outline

The talk will consist of two parts. First, we will show some theoretical properties and observations regarding interval pseudoinverse matrices. In the second part, we will focus on how to compute an enclosure of the interval pseudoinverse matrix for a given interval matrix. So far, there is only one known approach to this problem – Saraev's [2] interval extension of the standard algorithm for real matrices by Greville [1]. We shall compare this algorithm with our approach utilizing the existing methods for computing enclosures of eigenvalues of interval matrices.

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Interval matrices with Monge property

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Keywords: Monge property, interval analysis, interval optimization

Introduction

Monge matrices can be defined as follows.

Definition 1. *Let $M \in \mathbb{R}^{m\times n}$. Then M is a Monge matrix if*

$$m_{ij} + m_{k\ell} \leq m_{i\ell} + m_{kj}$$

for $1 \leq i < k \leq m, 1 \leq j < \ell \leq n$.

Monge property proved itself useful when it comes to many optimization problems [1]. The presence of Monge matrices in NP-hard problems as the travelling salesman problem results in polynomial algorithms [2]. For many other problems there are known asymptotical speed ups as well. We generalize the Monge property for interval matrices.

Interval generalization of Monge property

For interval generalization we defined two classes of interval matrices where for the first one all matrix realizations have to be Monge and for the other one at least one of the realizations has to be Monge.

Definition 2. *An interval matrix $\mathbf{M}$ is a weakly Monge if $\exists M \in \mathbf{M}$ such that M is Monge.*

Definition 3. *An interval matrix $\mathbf{M}$ is a strongly Monge if $\forall M \in \mathbf{M}$ it holds that M is Monge.*

Our results

The condition of interval strongly Monge matrix can be reduced into polynomially many conditions. The matrix can be also reduced into two realisations of the matrix. There is also a correspondence between strongly Monge matrices and a class of submodular functions on lattices. For interval weakly Monge matrices the analysis is more challenging, however, we proved the recognition is polynomial. We also studied closure properties of interval Monge matrices.

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Strong singularity of interval matrices

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Keywords: interval matrix, strong singularity, AE regularity

Introduction

One of the main properties of matrices with interval coefficients addressed in interval linear algebra is regularity [1, 2]. An interval matrix $\mathbf{A} \in \mathbb{IR}^{n\times n}$ is said to be *regular*, if all real matrices $A \in \mathbf{A}$ are non-singular, otherwise it is called *singular*.

Recently, the generalized property of *AE regularity* was proposed [3], motivated by the AE solutions of interval linear systems based on universal and existential quantifiers associated with each coefficient of the matrix. We consider a special case of AE regularity called *strong singularity*, which is the problem of checking whether each matrix $A \in \mathbf{A}$ is singular, or whether there is at least one non-singular matrix.

Strong singularity

A finite characterization of strong singularity was given by Hladík [3] by reducing the interval problem to the finite set of matrices $A \in \mathbf{A}$ with $a_{ij} \in \{\underline{a}_{ij}, \overline{a}_{ij}\}$ for each $i, j \in \{1, \ldots, n\}$, so-called *vertex matrices*.

Proposition 1. *An interval matrix* $\mathbf{A}$ *is strongly singular if and only if each vertex matrix is singular.*

However, other problems, such as determining the computational complexity or formulating a simpler characterization, remain open.

The talk

While regularity of interval matrices is quite well-studied, strong singularity still poses many questions, perhaps due to the fact that strongly singular matrices are rather rare. Nevertheless, a better understanding of strong singularity may contribute to the study of other more general types of AE regularity.

In this talk, we will link strong singularity to an interesting topic of interval linear programming, known as weak optimality, which is the problem of checking whether the set of optimal solutions over the scenarios of an interval program is non-empty. Furthermore, we will address the question of characterizing strongly singular matrices and investigate the property with respect to some special classes of interval matrices.

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Complexity of computing powers for various classes of interval matrices

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Keywords: interval matrix, matrix power, special matrix, NP-hardness

We consider the problem to compute a power of an interval matrix, which was shown to be NP-complete even when computing powers of degree 3 and higher [3]. We are studying various ways to handle the computation of interval matrix powers motivated by approaches for regularity radius problem [1, 2].

For an interval matrix $\mathbf{A}$, defined as $\mathbf{A} := \{A \in \mathbb{R}^{m\times n}; \underline{A} \leq A \leq \overline{A}\}$, we define k^{th} *power* as $\mathbf{A}^k := \{A^k : A \in \mathbf{A}\}$, that is, the set of k^{th} powers of all instances. This is not an interval matrix, so the goal is to compute a tight interval enclosure of $\mathbf{A}^k$. Recently, this problem has been shown to be polynomially tractable for a class of mixed interval and real valued matrices having intervals on the diagonal while leaving remaining elements to be real [4]. Motivated by this result we explore the problem for other specific classes of more general types aiming to draw a borderline of tractability. Motivated by the results for the regularity radius problem, one of the natural candidates is an interval tridiagonal matrix, which is the matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ having only the following elements as non-degenerate non-zero intervals: $a_{n,n}, a_{i,i}, a_{i,i+1}, a_{i+1,i}$ for all $i = 1, \ldots, n-1$.

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A survey on the interval matrix properties related to the linear complementarity problem

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Keywords: interval matrix, special matrix, linear complementarity problem, robustness

Introduction

We consider the linear complemetarity problem (LCP) [1], [4]

$$\begin{aligned} y &= Az + q, \quad y, z \geq 0, \\ y^T z &= 0, \end{aligned}$$

in which the entries of matrix A vary in given compact intervals. In other words, A comes from an interval matrix $\mathbf{A}$.

Many properties of the LCP are reflected by the properties of the constraint matrix A. There are known many classes of matrices having different properties with respect to LCP. In order that the problem has desired properties for each realization of interval data, we have to check the corresponding properties of interval matrix $\mathbf{A}$.

Matrix properties

In particular, we discuss the following matrix classes (another classes were addressed, e.g., in [2], [3]):

- *Copositive matrix:* $x^T Ax \geq 0$ for each $x \geq 0$. It ensures that the complementary pivot algorithm for solving the LCP works.

- *Semimonotone matrix:* The LCP has a unique solution for each $q > 0$.
- *Principally nondegenerate matrix:* It satisfies that the LCP has finitely many solutions for every $q \in \mathbb{R}^n$.
- *Column sufficient matrix:* It satisfies that for any $q \in \mathbb{R}^n$ the solution set of the LCP is a convex set.
- R_0*-matrix:* It satisfies that for any $q \in \mathbb{R}^n$ the LCP has a bounded solution set.

We characterize the interval version of the properties and also suggest several efficiently recognizable subclasses.

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Interval linear systems with parameters – branch and bound approach

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Keywords: parameters, interval linear systems, branch and bound, contractors

Introduction

In many applications (e.g., electrical engineering, structural mechanics or robotics) coefficients of a linear system depend on each other. Hence perturbation of one coefficient influences the value of the other ones. When interval uncertainty is introduced into parameters of a linear system (e.g., due to rounding errors or uncertainty in measurement) we obtain a parametric interval linear system. Let us have a k-dimensional vector of interval parameters $\boldsymbol{p} = [\underline{p}, \overline{p}] = \{p \in \mathbb{R}^k \mid \underline{p} \leq p \leq \overline{p}\}$. Then a parametric interval linear system is formally defined using $A^i \in \mathbb{R}^{m\times n}$, $b^i \in \mathbb{R}^m$, $i = 0, \ldots, k$ as

$$A(p)x = b(p), \quad p \in \boldsymbol{p},$$

where

$$A(p) = A^0 + \sum_{i=1}^{k} A^i p_i, \quad b(p) = b^0 + \sum_{i=1}^{k} b^i p_i.$$

A solution set of such a parametric system is defined as

$$\Sigma^p = \{x \in \mathbb{R}^n \mid A(p)x = b(p) \text{ for some } p \in \boldsymbol{p}\}.$$

Unlike in the case of plain interval linear systems a solution set does not need to be convex in each orthant, hence it might be even more challenging to capture its shape. Usually, such a solution set is enclosed by an n-dimensional box. For this purpose there exist many methods by various authors, e.g., [1], [2], [3].

However, in some cases such a description might be too crude to tell us more about the shape of the set Σ^p. Here, we are going to propose another approach known from CSP solving – branch and bound approach. For this purpose we design and discuss various contractors both on variable space and parameter space of a parametric interval linear system. Branch and bound methods can produce large number of boxes hence we are also going to address the issue of reducing the number of boxes. Such methods will be applicable also to interval nonlinear systems.

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Contributed talks

Further results involving positive semidefinite block matrices

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Keywords: singular values, operator norm, numerical radius, block matrices

Introduction

Let $\mathbb{M}_n(\mathbb{C})$ denote the space of $n \times n$ complex matrices. For Hermitian matrices $A, B \in \mathbb{M}_n(\mathbb{C})$, we write $A \geq B$ to mean $A - B$ is positive semidefinite. For $A \in \mathbb{M}_n(\mathbb{C})$, the singular values of A denoted by $s_1(A), s_2(A), \ldots, s_n(A)$ are the eigenvalues of the positive semidefinite matrix $|A| = (A^*A)^{\frac{1}{2}}$ enumerated as $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A)$ and repeated according to multiplicity. If A has real eigenvalues, then we label them as $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$.

It follows by Weyl's monotonicity principle [4], that if A, B are Hermitian and $A \leq B$, then $\lambda_i(A) \geq \lambda_j(A)$ for $j = 1, \ldots, n$.

For $A \in \mathbb{M}_n(\mathbb{C})$, let $\|A\|$ and $w(A)$ denote the operator norm and the numerical radius of A, respectively. It is known that $w(.)$ defines a vector norm on $\mathbb{M}_n(\mathbb{C})$, which is equivalent to the operator norm $\|.\|$. In fact for $A \in \mathbb{M}_n(\mathbb{C})$,

$$\frac{1}{2}\|A\| \leq w(A) \leq \|A\| \tag{1}$$

and if A is normal, then $w(A) = \|A\|$. Moreover, $w(A) = w(A^*)$ for any $A \in \mathbb{M}_n(\mathbb{C})$ and the norm $w(.)$ is weakly unitarily invariant (i.e, $w(A) = w(VAV^*)$ for any $A \in \mathbb{M}_n(\mathbb{C})$) and any unitary matrix $V \in \mathbb{M}_n(\mathbb{C})$).

Block matrices arise naturally in many aspects of matrix theory . The matrix $\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$ is called the *diagonal part* of $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ and $\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$ is the off-diagonal part.

A singular value inequality due to Tao [2] says that if $A, B, C \in \mathbb{M}_n(\mathbb{C})$ are such that $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$ is positive semidefinite, then

$$s_j(B) \leq s_j\left(\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}\right) \tag{2}$$

for $j = 1, \ldots, n$.

A related inequality due to Audeh and Kittaneh [3] says that if $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$ is positive semidefinite, then

$$s_j(B) \leq s_j(A \oplus B) \tag{3}$$

for $j = 1, \ldots, n$.

Recently, Burqan and Kittaneh [6] proved that if $A, B, C \in \mathbb{M}_n(\mathbb{C})$ are such that $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$ is positive semidefinite, then

$$s_j(B + B^*) \leq s_j((A + C) \oplus (A + C)) \text{ for } j = 1, \ldots, n \tag{4}$$

and

$$\|B + B^*\| \leq \|A + C\|. \tag{5}$$

On the other hand, Bourin et al. have proved in [5] that if $A, B, C \in \mathbb{M}_n(\mathbb{C})$ are such that $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$ is positive semidefinite, then

$$\left\|\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}\right\| \leq \|A + C\| + 2w(B)$$

in case of the operator norm. Several results about positive semidefinite block matrices can be found in [4] and [7].

The main purpose of this paper is to establish singular value inequalities related to the entries of block matrices. Upper bounds for the operator norms and the numerical radii of the off-diagonal part of block matrices are also given.

Basic properties

To establish and prove our results, we used matrix-valued inequalities associated with positive semidefinite block matrices which can be found in [1], [4], [8], and [9].

Main results

Theorem 1. *Let* $A, B, C \in \mathbb{M}_n(\mathbb{C})$ *are such that* $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$ *is positive semidefinite, then*

(a) *If* A *is positive definite, then* $s_j(A^{1/2}BA^{1/2}) \leq s_j(C^{1/2}A^{1/2})$ *for* $j = 1, \ldots, n$.

(b) *If* C *is positive definite, then* $s_j(C^{1/2}BC^{1/2}) \leq s_j(C^{1/2}A^{1/2})$ *for* $j = 1, \ldots, n$.

Theorem 2. *Let* $A, B, C \in \mathbb{M}_n(\mathbb{C})$ *are such that* $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$ *is positive semidefinite, then*

(a) *If* A *is positive definite, then* $s_j(A^{-1/2}B) \leq s_j(C^{1/2})$ *for* $j = 1, \ldots, n$.

(b) *If* C *is positive definite, then* $s_j(BC^{-1/2}) \leq s_j(A^{1/2})$ *for* $j = 1, \ldots, n$.

In what remains, we establish new upper bounds of the operator norm and the numerical radius of the off-diagonal parts of the positive semidefinite block matrices.

Theorem 3. *Let $A, B, C \in \mathbb{M}_n(\mathbb{C})$ are such that $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$ is positive semidefinite, then*

$$\|B\| \leq \max \|A\|, \|C\|.$$

Another upper bound for the operator norm of B in the block matrix $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$ is given in the following theorem:

Theorem 4. *Let $A, B, C \in \mathbb{M}_n(\mathbb{C})$ are such that $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$ is positive semidefinite, then*

$$\|B\|^2 \leq \min \left\{ \left\| \frac{C + B^*AB}{2} \right\|, \left\| \frac{C + BCB^*}{2} \right\| \right\}.$$

In the following theorem, we obtain a new estimate for the numerical radius of the off-diagonal parts of the positive semidefinite block matrices based on the operator norm of diagonal parts.

Theorem 5. *Let $A, B, C \in \mathbb{M}_n(\mathbb{C})$ are such that $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$ is positive semidefinite, then*

$$w(B) \leq \frac{1}{2} \|A + C\| .$$

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A perturbation result for distance matrices of trees with matrix weights

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Keywords: matrix weights, trees, Laplacian matrices, distance matrices

Introduction

A tree is a connected acyclic graph. Let $\mathcal{T} = (V, E)$ be a tree. We denote V by $\{1, \ldots, n\}$ and the elements of E by (i, j). To each edge (i, j), we assign a matrix weight W_{ij}, a real symmetric positive definite matrix of some fixed order, say, s. We now call $\mathcal{T}$ a weighted tree. The distance D_{ij} between the vertices i and j is the sum of all weights in the shortest path connecting i and j. Define

$$V_{ij} := -W_{ij}^{-1}, \; A_{ij} := \begin{cases} V_{ij} & (i,j) \in E \\ 0 & \text{otherwise} \end{cases} \text{ and } \; S_{ii} := \sum_j V_{ij}.$$

Let $L_{ij} := S_{ii} - A_{ij}$. We now define the distance and Laplacian matrices of $\mathcal{T}$ by $D := [[D_{ij}]]$ and $L := [[L_{ij}]]$, which are symmetric matrices of order ns. Let I_s be the identity matrix of order s. Define $U := \mathbf{1} \otimes I_s$, where $\mathbf{1}$ is the vector $(1, \ldots, 1)'$ in $\mathbb{R}^n$. Let δ_i be the degree of the i^{th} vertex of T. The following identity is a far reaching generalization of a remarkable formula of Graham and Lovász [4] obtained in Balaji and Bapat [1].

$$D^{-1} = -\frac{1}{2}L + \frac{1}{2}\Delta R^{-1}\Delta, \tag{1}$$

where $\tau := (2-\delta_1, \ldots, 2-\delta_n)'$, $R := \sum_{i,j} W_{ij}$ and $\Delta := \tau \otimes I_s$. Equation (1) leads to

$$(D^{-1} - L)^{-1} = \frac{1}{3}D + \frac{2}{3}U(U'D^{-1}U)^{-1}U'.$$

It can be shown that $U'D^{-1}U$ is a positive definite matrix. Thus every $(i,j)^{\text{th}}$ block of $(D^{-1}-L)^{-1}$ is a positive definite matrix. Motivated by this, we computed $(D^{-1}-S)^{-1}$, where S and D are Laplacian and distance matrices of two arbitrary trees with n-vertices. Surprisingly, from all the numerical examples, we found that each block in $(D^{-1}-S)^{-1}$ is positive definite. We shall precisely prove the following result by computing the inertia of $D^{-1}-S$, and using interlacing property and a nullity theorem of Fiedler and Markham [3]. This result also generalizes a known result in Bapat, Kirkland and Neuman [2].

Proposition 1. *Let T_1 and T_2 be any two trees with same number of vertices. Let D be the distance matrix of T_1 and L be the Laplacian of T_2. Then, $F := (D^{-1} - L) = [F_{ij}]$ is non-singular and each F_{ij} is positive definite, that is $F_{ij} + F_{ji}$ is positive definite.*

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Resistance matrices of balanced directed graphs

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Keywords: strongly connected balanced directed graphs, Laplacian matrix, Moore-Penrose inverse

Introduction

Let $G = (V, E)$ be a directed graph. The adjacency matrix of G is the $n \times n$ matrix $A := [a_{ij}]$ such that $a_{ij} = 1$ if $\overrightarrow{(i,j)} \in E$ and 0 otherwise. If for every i, the i^{th} row sum and the i^{th} column sum of A are equal, then we say that G is balanced. The Laplacian matrix of G is $L := D - A$, where A is the adjacency matrix of G and D is the diagonal matrix such that every row sum and column sum of L is zero.

Resistance between two vertices

Let $L^\dagger$ be the Moore-Penrose inverse of L. If i and j are any two vertices of G, then the resistance between i and j is defined by

$$r_{ij} := l^\dagger_{ii} + l^\dagger_{jj} - 2l^\dagger_{ij}. \tag{1}$$

It can be seen that this definition extends the usual definition of resistance distance if each edge in an undirected is replaced by two oppositely oriented edges. See [1] for basic properties of resistance distance.

Main results

Proposition 1. *Let r_{ij} denote the resistance between two vertices i and j of a strongly connected balanced digraph G. Then, $r_{ij} \geq 0$. In addition, If i, j and k are any three vertices of G, then $r_{ik} \leq r_{ij} + r_{jk}$.*

We then consider the resistance matrix $R := [r_{ij}]$. We find a new formula to compute R^{-1}.

Proposition 2.

$$R^{-1} = -\frac{1}{2}L + \frac{1}{\tau' R \tau}(\tau(\tau' + 1' \operatorname{diag}(L^{\dagger})M)),$$

where $M = L - L^T$ and τ is the $n \times 1$ vector with i^{th} entry given by

$$\tau_i = 2 - \sum_{\overrightarrow{(i,j)}} r_{ij}.$$

Proposition 2 extends the following well-known result of Graham and Lovász [2].

Proposition 3. *Let T be a tree with $V(T) = \{1, \ldots, n\}$. Let d_{ij} be the length of the shortest path between vertices i and j, and L be the Laplacian of T. Set $D := [d_{ij}]$. Then,*

$$D^{-1} = -\frac{1}{2}L + \frac{1}{2(n-1)}\tau\tau',$$

where $\tau = (2 - \delta_1, \ldots, 2 - \delta_n)'$ and δ_i is the degree of the vertex i.

We also prove results about sum of cofactors in R, extending similar results in the undirected case.

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The Friendship Graphs are determined by the eigenvalues of their normalized Laplacian

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The talk consists of two examples of applications of linear algebra to graph theory. The first is the classical proof of the Friendship Theorem. In the second we consider the graphs F_{pq} that are obtained by joining a vertex to all vertices of p disjoint copies of K_q. The graphs F_{p2} are the friendship graphs. We show that the graphs F_{pq} are determined by their normalized Laplacian spectrum iff $q > 1$ or $q = 1$ and $p < 3$, so in particular the friendship graphs are determined by their spectrum. This is joint work with Chen, Chen, Liang and Zhang.

Perturbations of matrix polynomial linearizations

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Keywords: matrix polynomial, linearization, perturbation

Abstract

A number of computational problems for matrix polynomials are solved by passing to linearizations, in particular Fiedler linearizations. We present an algorithm that finds which perturbations of the matrix coefficients of a polynomial correspond to a given perturbation of the entire linearization pencil. These results should help us to solve various distance problems for matrix polynomials. For example, finding a singular matrix polynomial, or a polynomial with a particular structure, nearby a given matrix polynomial.

Linear mixed models with random effects with known dispersion parameters

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Keywords: dispersion parameters, inference, linear mixed models

Introduction

There have been extensive studies in estimation in linear mixed models. However, the estimation of variance components in these models is not completely straightforward, even in the balanced case.

In this talk we will consider linear mixed models in which the random vectors have known dispersion parameters and second order moments but the relation matrices may not commute.

The goal is to present a procedure to estimate the variance components and the remaining estimable vectors in balanced or unbalanced models, assuming, or not, the normality. Besides this, we show how to construct the corresponding confidence regions and through duality, how to test hypotheses. A numerical example is provided in order to illustrate the theory.

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An overview of the estimators in additive models

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Keywords: ANOVA, cumulants, linear models, moments

Introduction

In our presentation we will consider additive models and will show how to use the cumulants to perform parameter estimation in such models. We also show how to generalize the Orthogonal Block Structure class of models, descarding the assumption of commutativity. We shall see that when we have a pair of independent and identical distributed models, for each treatment of a base design, we can evaluate the influence of the effects factors, on the second, third and fourth order central moments, as well as on the estimable functions.
We will present single and multiple additive models. In the last ones, we will have an additive model and, using the linear structure of these models, we will obtain homoscedastic vectors estimators which allow us to apply ANOVA and related techniques.

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Computation of min-plus eigenvalues via ultradiscrete integrable systems

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Keywords: min-plus algebra, eigenvalue, ultradiscrete hungry Lotka-Volterra system, ultradiscrete hungry Toda equation

Introduction

The qd algorithm is for computing eigenvalues of tridiagonal matrices and its recursion formula is known as the integrable discrete Toda equation. Our recent study [1], [2] shows that the time evolution of the integrable ultradiscrete Toda equation computes eigenvalues of tridiagonal matrices over min-plus algebra, where min-plus algebra is a semiring with two binary operations: $\oplus := \min$ and $\otimes := +$. It is known that eigenvalues over min-plus algebra coincide with the minimum value of average weights of circuits in the corresponding digraph [3]. The ultradiscrete hungry Toda equation

$$\begin{cases} Q_k^{(n+1)} = \bigotimes_{j=1}^{k} Q_j^{(n)} \oslash \bigotimes_{j=1}^{k-1} Q_j^{(n+1)} \oplus E_k^{(n)}, \quad k = 1, 2, \ldots, m, \\ E_k^{(n+1)} = Q_{k+1}^{(n)} \otimes E_k^{(n)} \oslash Q_k^{(n+1)}, \quad k = 1, 2, \ldots, m-1, \\ E_0^{(n)} := \infty, \quad E_m^{(n)} := \infty, \quad n = 0, 1, \ldots. \end{cases}$$

and the ultradiscrete hungry Lotka-Volterra system

$$\begin{cases} U_k^{(n+1)} = U_k^{(n)} \otimes \bigotimes_{j=1}^{M} \{(0 \oplus \Delta^{(n)} \otimes U_{k+j}^{(n)}) \oslash (0 \oplus \Delta^{(n+1)} \otimes U_{k-j}^{(n+1)})\}, \\ \qquad\qquad k = 1, 2, \ldots, M_m + M, \quad M_k := (M+1)k - M, \\ U_0^{(n)} := \infty, \quad U_{M_m+M+1}^{(n)} := \infty \end{cases}$$

are generalizations of the ultradiscrete Toda equation and ultradiscrete Lotka-Volterra system, respectively. In this talk, we show that these integrable systems can compute eigenvalues of banded matrices over min-plus algebra. To be more precise, eigenvalues of the intended matrices are conserved quantities of the integrable systems and a variable converges to an eigenvalue by discrete time evolution of the integrable systems. The resulting algorithm is a min-plus analogue of a generalization of the qd algorithm and the dLV algorithm.

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Bimatrix games over proper cones

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Keywords: bimatrix games, proper cone, non-linear programming problem

Abstract

Bimatrix game is the game played by the two players(player I and II) whose payoff matrices are $A_{m\times n}$ and $B_{m\times n}$ respectively. In detail, If player I(row player) choose to play by i^{th} row and player II(column player) by j^{th} column, their expected payoff is a_{ij} and b_{ij} respectively. In case players play with probability distributions x^* on the rows and y^* on the columns, then player I and II's expected payoff is $\langle x^*, Ay^*\rangle$ and $\langle x^*, By^*\rangle$, respectively. If the following inequalities are true for all probability vectors $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$,

$$v_1 := \langle x^*, Ay^*\rangle \geq \langle x, Ay^*\rangle,$$
$$v_2 := \langle x^*, By^*\rangle \geq \langle x^*, By\rangle.$$

we say (x^*, y^*) forms a equilibrium pair. Payoff v_1 and v_2 at the equilibrium pair is called the value of the players I and II, respectively. In this talk we present the generalization concepts of value and Nash equilibrium of the bimatrix game. And we extend some of the results of Raghavan to this general case. Also we show the equilibrium equivalence with the solution of some non-linear programming problem.

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EBLUP: Empirical Best Linear Unbiased Predictors

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Keywords: best linear unbiased estimator, BLUE, best linear unbiased predictor, BLUP, empirical best linear unbiased estimator, EBLUE, empirical best linear unbiased predictor, EBLUP, linear model, linear model with new observations, linear mixed model prediction mean squared error.

Abstract

We outline the basic properties of empirical best linear unbiased predictors (EBLUPs) and discuss some of the issues that arise in estimating their prediction mean squared errors. We introduce EBLUPs in the context of the linear mixed model with unknown covariances and briefly describe some of their applications. We then consider their particular application to small area estimation and outline ways to estimate the prediction mean squared error that have been developed in this context.

Complexity of powering real large matrices and small polynomials

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Keywords: reliable computing, bit-complexity, exponential powering

Motivation and summary of results

Motivated by difference schemes for solving partial differential equations approximately up to error $1/2^n$, we consider [2] the problem of powering exponentially large real matrices $\mathbb{R}^{2^n \times 2^n} \ni A \mapsto A^{2^n}$. These cannot be input/output in polynomial time but

i) have fixed entries; more formally, $A_{I,J} = a_{M(I,J)}$ for some polynomial-time computable sequence $a_M \in \mathbb{R}$ and $M : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$.

ii) have bounded powers $\forall K : \|A^K\| \leq 1$, i.e., do not 'blow up' (which again would trivially prevent polynomial-time computability).

iii) ask for approximating, given indices $I, J \in \{0, \ldots, 2^n - 1\}$ in binary, the (I, J)-th entry of A^{2^n} up to absolute error $1/2^n$.

Note that (ii) renders the problem trivial in the traditional integer setting yet new in the real case. 'Padding' A makes it no loss of generality to combine precision and dimension in one parameter n.

Theorem 1. a) *The above matrix powering problem can be solved in space (=memory) polynomial in n, that is, belongs to the real complexity class* $\mathbb{R}$PSPACE [1, §7.2.1]. b) *In general it cannot be solved in time polynomial in n unless* PSPACE=NP=P.

a) follows from repeated squaring and analysis of approximation error propagation. For b), encode the configuration graph of any polynomial-space Turing machine as exponential-size adjacency matrix.
Difference schemes often have structure, known to improve complexity from quadratic to near-linear in the dimension [3] but still exponential in n. Circulant matrices of constant bandwidth arise from PDEs with periodic boundary conditions and (i') can be input in polynomial time.

Theorem 2. a) *The 2^n-th power of a given circulant matrix of bandwidth two according to* (i')+(ii)+(iii) *is computable in time* $\mathrm{poly}(n)$.

a') *Coefficients of p^{2^n} for given linear polynomial $p = a+bX$ of 1-norm $\|p\| = |a|+|b| \le 1$ can be computed in time polynomial in n.*

b) *For any constant bandwidth and fixed circulant matrix with polynomial-time computable entries* (i)+(ii)+(iii) *can be computed in the real complexity class $\mathbb{R}\#\mathsf{P}$.*

b') *The coefficients of p^{2^n}, for a given polynomial p of constant degree with $\|p\| \le 1$, can be computed in the real complexity class $\mathbb{R}\#\mathsf{P}$.*

Regarding (a') note that the K-the coefficient of $(a + Xb)^N$ is $\binom{N}{K} \cdot a^K \cdot b^{N-K}$ and bounded by 1, but $\binom{2^n}{2^n/2}$ itself has exponential bitlength and thus cannot be computed in polynomial time. Problems (a) and (a') are actually equivalent; same for (b) and (b').

We conjecture that (b) and (b') can be improved to polynomial time. In case $p = (1 + bX + cX^2)/(1 + b + c)$ with $a, b, c > 0$ and $b^2 \neq 4c$, the 2^n-th coefficient $T_{2^n}(b, c)/(1+b+c)^{2^n}$ of p^{2^n} is polynomial-time computable according to the following asymptotic expansion [4]:

$$T_N(b,c) = \frac{(b+2\sqrt{c})^{N+1/2}}{2\sqrt[4]{c}\sqrt{N\pi}} \cdot \left(1 + \frac{b-4\sqrt{c}}{16N\sqrt{c}} + \frac{(3b-4\sqrt{c})^2}{512cN^2} + \mathcal{O}\big(1/N^3\big)\right)$$

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Generalized eigenvectors of bounded linear operators in a separable Hilbert space

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We define the eigenvector corresponding to pseudospectrum and the eigenvector corresponding to condition spectrum of a bounded linear operator in a separable Hilbert space. The relation connecting approximate eigenvalues, pseudo eigenvectors and condition eigenvectors are found. The sufficient condition for a bounded linear operator in a separable Hilbert space to possess an almost invariant subset is found. We prove the existance of common eigenvector corresponding to the pseudospectrum and common eigenvector corresponding to the condition spectrum for various class of almost commuting bounded linear operators in a separable Hilbert space. The results are also illustrated with some operators.

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Discrete Börg-type theorems

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Börg theorem is a classical result in inverse spectral theory. It was proved by G. Börg in 1946. It states that the periodic potential of one dimensional Schrödinger operator is constant almost everywhere if and only if the spectrum of the operator is connected. The connection of Börg theorem with the density of a string and other important problems in Physics were observed later in 1960′s (Recall the popular article "Can One hear the Shape of a Drum?" by M. Kac [6]). The discrete version and generalization of the result were also known since 1975 (see [1] for eg.). Such results are referred as Börg-type theorems. In [2], we proved the Börg-type theorem for discrete periodic Schrödinger operators, identifying them as block Toeplitz operators. In [4], these results were extended to much general block Toeplitz operators.

Recently in [5], we considered the case when the potential is non constant, and obtained some estimates of size of the spectral gaps. A better lower bound is obtained in [3] very recently. An open problem in this regard is to discover the relation between number of spectral gaps and the essential period. Also, the pseudo-spectral version for the non normal Schrödinger operators is another important area of research. We wish to discuss the recent developments in this regard.

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Prescribing convergence behavior of block Arnoldi and GMRES

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Keywords: block Krylov subspace methods, multiple right-hand sides, block GMRES, convergence, spectrum, block companion matrix

Abstract

Block Krylov subspace methods are iterative methods for solving systems of linear equations with multiple right-hand sides. At each step, all solutions to the system are sought in the space containing the contribution of each individual right-hand side, which significantly reduces the iteration count compared to solving the systems sequentially (one by one). We focus on block methods for non-Hermitian systems, in particular on block GMRES. While it is known that any non-increasing convergence curve is possible for standard GMRES with one right-hand side and a matrix with a given spectrum [1], no analog of this result is currently available for block methods, when multiple systems are solved at once. Using a recent framework for studying these methods as being a single linear system over a *-ring of complex matrices [2], we develop convergence results for block Arnoldi and block GMRES. In particular, we show what convergence behavior is admissible for block GMRES and how the matrices and right-hand sides producing such behavior can be constructed. Moreover, we show that the convergence of the block Arnoldi method for eigenvalue approximation can

be fully independent of the convergence of block GMRES for the same coefficient matrix and the same starting vectors.

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Matrix approximation via entropy loss function

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Keywords: approximation, compound symmetry, entropy loss function, estimation, first-order autoregression, Kronecker product, regularization

Abstract

In this paper, we focus on determining the best approximation of a positive definite symmetric matrix by a matrix having a special structure. In particular, we consider the Kronecker product structure with both components unstructured, or with one component structured as compound symmetry or the first-order autoregression, with the use of the entropy loss as a measure of discrepancy. We show some properties of the entropy loss function and we prove that in all the cases the approximation preserves the positive definiteness property.

Presented results can be widely used in multivariate statistics, for example for regularizing the covariance structure of a given covariance matrix or in testing hypotheses about the covariance structures. Simulation studies show that the proposed approach is reliable in the mentioned issues.

Eigenvalue localization under partial spectral information

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Keywords: bound, eigenvalue, inclusion, Gershgorin, Ostrowski, Brauer

Introduction

Eigenvalue inclusion regions for matrices have been studied for a long time, but inclusion regions that take into consideration spectral properties of the eigenspaces or eigenvectors seem to be somewhat less commonly encountered. Examples of the latter can be found in [1] and [3], where nonsimple eigenvalues are considered, and [4], which is mainly devoted to upper bounds on the subdominant eigenvalues of a nonnegative matrix when the dominant eigenvalue is known. Here we investigate how a computed eigenpair can be used to improve well-known inclusion regions for the remaining eigenvalues, and show the relation of this process to eigenvalue deflation techniques. The results are applied to nonnegative and real symmetric matrices and illustrated by numerical examples.

Main results

Theorem 1. *Let $A \in \mathbb{C}^{n\times n}$, let (μ, w) be an eigenpair for A, and let $z \in \mathbb{C}^n$ be arbitrary. Then any eigenvalue of A that is different from μ lies in the union*

$$\bigcup_{i=1}^{n} \left\{ \lambda \in \mathbb{C} : |\lambda - a_{ii} + z_i w_i| \le \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ji} - z_i w_j| \right\}.$$

Theorem 2. *Let $A \in \mathbb{C}^{n\times n}$, let (μ, w) be an eigenpair for A, and let $z \in \mathbb{C}^n$ be arbitrary. Then any eigenvalue of A that is different from μ lies in the union*

$$\bigcup_{\substack{i,j=1\\ i<j}}^{n} \{\lambda \in \mathbb{C} : |\lambda - a_{ii} + z_i w_i|\,|\lambda - a_{jj} + z_j w_j| \leq \left(\sum_{\substack{k=1\\ k\neq i}}^{n} |a_{ki} - z_i w_k| \right) \left(\sum_{\substack{k=1\\ k\neq j}}^{n} |a_{kj} - z_j w_k| \right) \Bigg\} .$$

We show how to minimize the right-hand sides of the inequalities defining the inclusion regions, which are, in fact, Fermat-Weber problems, for symmetric and for positive matrices. The ovals of Cassini appearing in Theorem 2 can easily be circumscribed by rectangles that are more convenient. We illustrate our results with extensive numerical experiments.

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On the vectors associated with the roots of max-plus characteristic polynomials

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Keywords: max-plus algebra, eigenvalue, characteristic polynomial

Introduction

The max-plus (or tropical) algebra $\mathbb{R}_{\max} := \mathbb{R} \cup \{\varepsilon\}$, where $\varepsilon = -\infty$, is a semiring with addition $a \oplus b := \max\{a, b\}$ and multiplication $a \otimes b := a + b$. The max-plus eigenvalue problem is much different from the conventional one since the eigenvalues of a matrix do not coincide with the roots of its characteristic polynomial. In this talk, we give a notion of "algebraic eigenvectors" so that we can characterize all the roots. The adjective "algebraic" is taken from [1], in which the roots of characteristic polynomial are called algebraic eigenvalues.

Basic properties

Let $A \in \mathbb{R}_{\max}^{n \times n}$ be a square matrix. A scalar $\lambda \in \mathbb{R}_{\max}$ and a vector $\boldsymbol{v} \neq {}^t(\varepsilon, \ldots, \varepsilon)$ satisfying

$$A \otimes \boldsymbol{v} = \lambda \otimes \boldsymbol{v}$$

are called an eigenvalue and an eigenvector of A, respectively. The characteristic polynomial of A is defined by

$$\varphi_A(t) := \det(A \oplus t \otimes E) = \bigoplus_{\sigma \in S_n} \bigotimes_{i=1}^{n} (A \oplus t \otimes E)_{i\sigma(i)},$$

where E is the identity matrix and $(B)_{ij}$ means the (i,j) entry of matrix B. We can factorize $\varphi_A(t)$ into the product of linear forms:

$$\varphi_A(t) = (t \oplus \lambda_1)^{\otimes p_1} \otimes (t \oplus \lambda_2)^{\otimes p_2} \otimes \cdots \otimes (t \oplus \lambda_m)^{\otimes p_m}.$$

We call $\lambda_1, \lambda_2, \ldots, \lambda_m$ the roots of $\varphi_A(t)$. It is known that the maximum root coincides with the maximum eigenvalue of A, whereas other roots may not be eigenvalues [1].

Main results

For $A = (a_{ij}) \in \mathbb{R}_{\max}^{n\times n}$, assume that each term of $\varphi_A(t)$ is attained with exactly one permutation. Then we show that $\lambda \in \mathbb{R}_{\max}$ is a root of $\varphi_A(t)$ if and only if there exist a set $J \subset \{1, 2, \ldots, n\}$ of indices, a permutation π on J and a vector $\boldsymbol{v} \neq {}^t(\varepsilon, \ldots, \varepsilon)$ satisfying

$$(A_{\setminus\pi} \oplus E_J) \otimes \boldsymbol{v} = (A_\pi \oplus E_{\setminus J}) \otimes \boldsymbol{v}.$$

Here,

$$(A_\pi)_{ij} = \begin{cases} a_{ij} & \text{if } i \in J, j = \pi(i), \\ \varepsilon & \text{otherwise,} \end{cases} \quad (A_{\setminus\pi})_{ij} = \begin{cases} \varepsilon & \text{if } i \in J, j = \pi(i), \\ a_{ij} & \text{otherwise,} \end{cases}$$

$$(E_J)_{ij} = \begin{cases} 0 & \text{if } i = j \in J, \\ \varepsilon & \text{otherwise,} \end{cases} \quad \text{and} \quad (E_{\setminus J})_{ij} = \begin{cases} 0 & \text{if } i = j \notin J, \\ \varepsilon & \text{otherwise.} \end{cases}$$

We call such vector $\boldsymbol{v}$ an algebraic eigenvector of A with respect to λ.

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Evolution of Markov chains under transition matrix perturbation

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Keywords: Markov models, stochastic matrices, non-diagonalizable matrices, perturbation theory

Introduction

To determine the evolution of a Markov chain, the calculation of matrixpowers is essential. For the computation of matrixpowers of transition matrices, there is an important difference between diagonalizable and non-diagonalizable matrices. Under certain conditions[1], the evolution of both kinds of matrices is considered. This will be done by considering a suitable perturbation on the non-diagonalizable transition matrix, which preserves certain spectral properties.

Main results

Starting from a non-diagonalizable transition matrix A, we perform a suitable perturbation, such the perturbation preserves certain spectral properties. The preservation of spectral properties is expressed as conditions on the perturbation matrix. For a suitable perturbed matrix $\tilde{A}$, we find an upper bound for $\left\|\tilde{A}^k - A^k\right\|$. The convergence is studied and if the conditions are satisfied, the rate of convergence is found to be exponential, for k large. The behaviour of $\left\|\tilde{A}^k - A^k\right\|$ for small k, is also considered and explained. Furthermore, other consequences of the upper bound on Markov chains are derived.

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Recent development in the core problem theory

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Keywords: linear approximation problem, core problem, orthogonally invariant minimization

Introduction

Let us consider a linear approximation problem $Ax \approx b$, $b \notin \text{range}(A)$. It is well known that the related total least squares (TLS) minimization

$$\min \|[g, E]\|_F \quad \text{such that} \quad (b+g) \in \text{range}(A+E)$$

may not have a solution for the given (A, b). Paige and Strakoš [3] explained this phenomenon through the so-called core problem concept. They introduce an orthogonal trasnformation

$$(P^T A Q)(Q^T x) \equiv \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \approx \begin{bmatrix} b_1 \\ 0 \end{bmatrix} \equiv (P^T b)$$

that extracts from the original data a minimally dimensioned subproblem $A_{11}x_1 \approx b_1$ called the *core problem*. Among other properties, it always has the TLS solution that can be transformed back to the orignal problem $Ax \approx b$. In this way, it justifies, e.g., the *nongeneric approach* of Van Huffel and Vandewalle [4].

The matrix right-hand side case and beyond

The matrix right-hand side variant of the TLS minimization (see [4]), i.e., for $AX \approx B$, where $B \in \mathbb{R}^{m\times d}$, motivated a generalization of the core problem. However, the matrix variant of core problem still may not have a TLS solution which raises a natural question: "Why?"

In order to better understand the matrix case we show how to extend the concepts into

(i) tensor right-hand side problems $A \times_1 \mathcal{X} \approx \mathcal{B}$ [1], and to

(ii) problems with generalized models, in particular $A_{\mathfrak{L}} X A_{\mathfrak{R}} \approx B$ [2],

to see the matrix case in a wider context.

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Gradient neural network models for solving matrix equation $AXB = D$ in real time

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Keywords: gradient neural network; Zhang neural network; matrix inverse; dynamic equation; activation function

We consider the gradient neural network model for solving the matrix equation $AXB = D$ in real time. It is shown that the model is globally convergent to the general solution of the matrix equation, which can be determined by the choice of the initial matrix. Several applications on computing the matrix generalized inverses are also shown. The model is extended to the case when the gain parameter is function of time variable. All results are illustrated on different numerical examples.

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Lanczos-like method for the time-ordered exponential

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Keywords: time-ordering, matrix differential equations, ordered exponential, Lanczos algorithm

Introduction

Let $\mathsf{A}(t)$ be a time-dependent matrix with t in an interval I. The time-ordered exponential of $\mathsf{A}(t)$ is defined as the unique solution $\mathsf{U}(t)$ of the system of coupled linear differential equations

$$\mathsf{A}(t)\mathsf{U}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathsf{U}(t),$$

with initial condition $\mathsf{U}(0) = I$. When A commutes with itself at all times, i.e., $\mathsf{A}(t)\mathsf{A}(t') - \mathsf{A}(t')\mathsf{A}(t) = 0$ for every $t, t' \in I$, then the ordered exponential is simply given by a matrix exponential as

$$\mathsf{U}(t) = \exp\left(\int_0^t \mathsf{A}(\tau)\, d\tau\right).$$

In the general case, however, when A does not commute with itself at all times, the ordered exponential has no known explicit form in terms of A.

The problem of evaluating $\mathsf{U}(t)$ is a central question in the field of system dynamics, in particular in quantum physics where A is the quantum Hamiltonian. Until now, few methods have been proposed to approximate the ordered exponential, but a satisfactory answer to this problem is still missing. In [1], Giscard et al. proposed a method to obtain ordered exponentials using graph theory and necessitating only the entries $\mathsf{A}(t)$ to be bounded functions of time. While this approach provides exact solutions and is convergent, it suffers from a computational drawback. We will introduce a model-reduction strategy that solves such computational issue by a Lanczos-like algorithm, giving a converging and computable (in term of complexity) strategy for getting $\mathsf{U}(t)$. Such a technique is derived by extending to the time-depending settings the well-known connections between the Lanczos algorithm, the moment problem, graph approximations, and continued fractions.

Main results

Let us define the convolution-like $*$ product between $\mathsf{A}_1(t',t)$ and $\mathsf{A}_2(t',t)$ as

$$\big(\mathsf{A}_2 * \mathsf{A}_1\big)(t',t) := \int_t^{t'} \mathsf{A}_2(t',\tau)\mathsf{A}_1(\tau,t)\,\mathrm{d}\tau.$$

In the spirit of existing Lanczos approaches for approximating matrix functions, given a time-dependent matrix $\mathsf{A}(t',t)$, we construct a matrix T_n of size n with a simpler (tridiagonal) structure and so that

$$(\mathsf{A}^{*j})_{k,\ell} = (\mathsf{T}_n^{*j})_{1,1}, \qquad \text{for} \qquad j = 0,\ldots,2n-1.$$

Then the solution can be expressed by the techniques in [1].

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Some further properties of the linear sufficiency in the linear model

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Keywords: BLUE, BLUP, linear sufficiency, linear model with new observations, transformed linear model

Introduction

In this talk our focus lies in the linear statistical model $y = Xb + e$, denoted as $M = \{y, Xb, V\}$, supplemented with the new unobservable random vector y_*, coming from $y_* = X_*b + e_*$. A linear statistic Fy is called linearly sufficient for estimable X_*b if there exists a matrix A such that AFy is the best linear unbiased estimator, BLUE, for X_*b. In matrix terms, this means that there exists a matrix A such that the equation

$$AF(X_* : VX^\perp) = (X_* : 0)$$

is satisfied. The concept of linear sufficiency with respect to a predictable random vector is defined in the corresponding way but considering the best linear unbiased predictor, BLUP, instead of BLUE. In this talk, we consider the linear sufficiency of Fy with respect to y_*, X_*b, and e_*. For some old and recent references in this area, see the References.

Main results

There is a strong connection between the linear sufficiency concept and certain properties of the transformed model $T = \{Fy, FXb, FVF'\}$. We will introduce necessary and sufficient conditions for the equality

of the multipliers of y providing BLUEs and BLUPs in the original model M and in the transformed model T.

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Partial sum of eigenvalues of random graphs

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Keywords: adjacency matrix, Erdős–Rényi model, random matrices

Introduction

Consider a graph G on n vertices and let A be its adjacency matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Here we investigate the partial sum $S_k(G) = \sum_{i=1}^{k} \lambda_i$ for $1 \leq k \leq n$. This parameter concerns theoretical chemistry in the Hückel molecular orbital theory. In this theory the behavior of the so-called π-electrons in an unsaturated conjugated molecule is described. If the carbon-atom skeleton of the underlying conjugated molecule is represented as a graph, then each eigenvalue of the adjacency matrix determines the energy level of a π-electron, and the sum S_k determines the total energy of these electrons. Furthermore, the corresponding eigenvector describes how the π-electron moves within the molecule, i.e., the molecular orbital.

Ivan Gutman introduced the the energy of a graph $E(G) = \sum_{i=1}^{n} |\lambda_i|$ which relates to S_k by $E(G) = 2 \max_{1 \leq k \leq n} S_k$. Since then the energy of a graph has been intensively investigated. For a survey and bounds on the energy see [3], [5], the papers [2], [6] for recent developments on the energy, and [1], [4] for bounds on S_k. Here, we present new bounds for the partial sum of the eigenvalues of random graphs.

Bounds for random graphs

Let $\{G_n\}$ be a sequence of graphs drawn from the Erdős–Rényi random graph model distribution $G(n, 1/2)$. We say that a property holds for

almost all graphs if the property holds for G_n with probability tending to 1 as $n \to \infty$.

Theorem 1. *For almost all graphs it holds* $s_k(G) \leq \frac{e(G)+k^2}{(0.99n)^{1/2}}$.

Let $s(t, G)$ be the sum of the adjacency eigenvalues of G larger or equal than $t\sqrt{n}$.

Theorem 2. *For almost all graphs it holds*

$$s(t, G) \leq \frac{2}{3\pi} n^{3/2} \left(\frac{e(G)}{\binom{n}{2}} + \frac{\left(1-t^2\right)^3}{2} + O(1) \right).$$

Acknowledgement

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On the structure-preserving diagonalization of Hamiltonian matrices

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Keywords: symplectic matrix, Hamiltonian matrix, skew-Hamiltonian matrix, indefinite inner product, matrix decomposition

We consider the indefinite inner product $[\cdot,\cdot] : \mathbb{C}^{2n} \times \mathbb{C}^{2n} \to \mathbb{C}$ defined by

$$[x, y] := x^H J_{2n} y, \quad \text{where } J_{2n} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}, \tag{1}$$

and matrices $A \in \mathbb{C}^{2n\times 2n}$ that posses special structure related to the above inner product. We give necessary and sufficient conditions for such matrices to admit a structure-preserving diagonalization and discuss further implications. These results are based on [1, Sec. 9-10].

The *adjoint* $A^\star$ of a matrix $A \in \mathbb{C}^{2n\times 2n}$ with respect to the inner product (1) is the uniquely defined matrix that satisfies

$$[Au, v] = [u, A^\star v] \quad \text{and} \quad [u, Av] = [A^\star u, v]$$

for all $u, v \in \mathbb{C}^{2n}$. It can be explicitly expressed as $A^\star = J_{2n}^T A^H J_{2n}$ and induces several important classes of matrices: a matrix $S \in \mathbb{C}^{2n\times 2n}$ for which it holds that $S^\star = S^{-1}$ is called *symplectic*. Moreover, a matrix $B \in \mathbb{C}^{2n\times 2n}$ is called *skew-Hamiltonian* whenever $B^\star = B$ and *Hamiltonian* if $B^\star = -B$. All three types of matrices belong to the class of J_{2n}-normal matrices which are defined through the relation $AA^\star = A^\star A$. Whenever some $A \in \mathbb{C}^{2n\times 2n}$ is diagonalizable via a similarity transformation $S^{-1}AS = S^\star AS$ where $S \in \mathbb{C}^{2n\times 2n}$ is symplectic, it is easily checked that A must be J_{2n}-normal. However:

Not every J_{2n}-normal matrix is diagonalizable and can be diagonalized by a symplectic similarity transformation.

Starting from the question what conditions need to be imposed on a J_{2n}-normal matrix $A \in \mathbb{C}^{2n\times 2n}$ to be symplectic diagonalizable, we present and discuss the following results through the example of Hamiltonian matrices:

- A (diagonalizable) Hamiltonian matrix $A \in \mathbb{C}^{2n\times 2n}$ is symplectic diagonalizable if and only if the following holds: Given any basis $v_1, \ldots, v_k \subset \mathbb{C}^{2n}$ of the eigenspace for any purely imaginary eigenvalue λ of A, the matrix

$$V^H J_{2n} V, \quad V = \begin{bmatrix} v_1 & \cdots & v_k \end{bmatrix} \in \mathbb{C}^{2n\times k}$$

  has equally many positive and negative imaginary eigenvalues.

- A Hamiltonian matrix $A \in \mathbb{C}^{2n\times 2n}$ with a complete set of orthogonal eigenvectors is symplectic diagonalizable if and only if it is diagonalizable by a unitary *and* symplectic matrix.

- The unitary and symplectic diagonalizability of a Hamiltonian matrix $A \in \mathbb{C}^{2n\times 2n}$ is equivalent to the existence of a decomposition

$$A = N - N^\star$$

  where $N \in \mathbb{C}^{2n\times 2n}$ satisfies $NN^\star = N^\star N = 0$. In this decomposition, $\operatorname{span}(N)$ is invariant for A and $u^H J_{2n} v = 0$ holds for any $u, v \in \operatorname{span}(N)$ (i.e. $\operatorname{span}(N)$ is isotropic).

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Complete multipartite graphs that are determined, up to switching, by their Seidel spectrum

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Keywords: complete multipartite graphs, Seidel matrix, Seidel switching, S-determined graphs

Introduction

With each simple graph G we may associate several matrices. Some common examples are the adjacency matrix, the Laplacian, the signless Laplacian, and the distance matrix. If X is such a matrix type, we say a graph is X*-determined* if every graph which is X-cospectral with G is isomorphic to G. For example, it is known that complete multipartite graphs are distance-determined, but not adjacency-determined, see [2], [3]. Two surveys on adjacency-cospectral graphs are [4], [5].

The *Seidel matrix* of G is $S(G) = J - I - 2A(G)$, where $A(G)$ is the adjacency matrix of G, I the identity matrix and J is the all-ones matrix. No graph on $n \geq 2$ vertices is Seidel-determined, since

any graph obtained from G by Seidel switching (see below) has the same Seidel spectrum. We consider G to be determined by its Seidel spectrum, up to switching, if any graph with the same spectrum is switching equivalent to a graph isomorphic to G. We consider here complete k-partite graphs: which of these graphs are Seidel-determined up to switching?

Basic properties

For a graph $G = (V(G), E(G))$ the Seidel matrix $S(G)$ is a symmetric matrix with zero diagonal and all other entries in $\{1, -1\}$. If $U, W \subseteq V(G)$ form a partition of $V(G)$, a *Seidel switching* with respect to U transforms G to a graph H by deleting the edges between U and W and adding an edge between vertices $u \in U$ and $w \in W$ if $(u, w) \notin E(G)$. Seidel switching is an equivalence relation and we say that G and H are *switching equivalent*. If H is obtained from G by Seidel switching, then $S(H) = \Lambda S(G)\Lambda$, where Λ is a signature matrix (a diagonal matrix with ± 1 diagonal entries — here 1's correspond to vertices of U and -1's to vertices of W). Hence $S(H)$ and $S(G)$ are similar and have the same spectrum, though the graphs G and H are generally not isomorphic. We say that a graph G is *Seidel determined, up to switching,* (or, in short, *S-determined*) if the only graphs with same Seidel spectrum are switching equivalent to a graph isomorphic to G.

Main results

We show that any graph which has the same spectrum as a complete k-partite graph is switching equivalent to a complete k-partite graph, and if the different partition sets sizes are $p_1, \ldots, p_l$, and there are at least three partition sets of each size p_i, $i = 1, \ldots, l$, then the complete multi-partite graph is S-determined. Sufficient conditions for a complete tripartite graph to be S-determined are discussed. The results appear in [1].

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Zeroing neural networks in linear algebra

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Keywords: Zhang neural network, generalized inverses, time-varying matrix

Abstract

We investigate various applications of dynamical systems in matrix computations. The dynamical system approach is a powerful tool for solving many kinds of matrix algebra problems because of its parallel nature, possibility to provide output within a predefined time in real-time applications, convenience of hardware implementation, global convergence without any condition, applicability to online computation with time-varying matrices.

The ZNN design is defined upon the usage of an appropriate matrix-valued error-monitoring Zhang function (Zf shortly), denoted by $E(t)$. The original ZNN design arising from the Zf $E(t)$ is defined as the dynamical system

$$\dot{E}(t) := \frac{\mathrm{d}E(t)}{\mathrm{d}t} = -\gamma \mathcal{H}(E(t)). \tag{1}$$

In (1), $\dot{E}(t)$ denotes the time derivative of $E(t)$, $\gamma \in \mathbb{R}$, $\gamma > 0$ is the scaling parameter, and $\mathcal{H}(\cdot) : \mathbb{C}^{n\times m} \mapsto \mathbb{C}^{n\times m}$ denotes a complex-valued elementwise applicable matrix-to-matrix activation function.

New stream in the research of zeroing neural design is the varying-parameter ZNN (VPZNN) design, defined by

$$\dot{E}(t) = -\gamma(t) \mathcal{H}(E(t)), \tag{2}$$

Essentially, the VPZNN design uses relatively small, time-varying values $\gamma(t)$ instead of "as large as possible" constant values γ.

We investigate the VPZNN design (2) arising from the Zf defined in [2], [3]

$$E_{F,G}(t) = (G(t)A(t) + \lambda I)\, X(t) - F(t), \tag{3}$$

where $A(t) \in \mathbb{C}_r^{m\times n}$ is a given $m \times n$ time-varying matrix of a constant rank r, $F(t) \in \mathbb{C}^{n\times m}$ be an arbitrary $n \times m$ matrix and $G(t) \in \mathbb{C}_s^{n\times m}$ is a selected $n \times m$ matrix of a constant rank $0 < s \leq r$. The Zf proposed in [1] can be obtained in the particular case $F(t) \equiv G(t)$.

Also, the following integration-enhanced noise-tolerant VPZNN model, called $IENTVPZNN(A, F, G)$, will be be considered:

$$\dot{E}_{F,G}(t) = -\gamma(t)E_{F,G}(t) - \zeta \int_0^t E_{F,G}(\tau)\mathrm{d}\tau + N(t), \tag{4}$$

where $N(t)$ is the noise in a matrix form.

In addition, a new design formula for $E_{F,G}(t)$ is proposed:

$$\frac{\mathrm{d}E_{F,G}(t)}{\mathrm{d}t} = -\gamma(t)\mathcal{H}\left(k_1 E_{F,G}(t) + k_2 E_{F,G}{}^{q/p}(t)\right), \tag{5}$$

where $k_1, k_2 > 0$ and p, q denote positive odd integers satisfying $p > q$.

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Characterization of half-radial matrices

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Keywords: field of values, numerical radius, Crouzeix's conjecture

Introduction

Numerical radius $r(A)$ is the radius of the smallest ball with the center at zero containing the field of values $W(A)$ of a given square matrix A. It is well-known that

$$r(A) \leq \|A\| \leq 2r(A)\,,$$

where $\|\cdot\|$ denotes the matrix 2-norm. Matrices attaining the lower bound are called radial, and have been analyzed thoroughly. In this presentation we investigate matrices satisfying $r(A) = \|A\|/2$, and call them *half-radial*. We summarize the existing results and formulate new ones. In particular, we investigate their singular value decomposition and algebraic structure. Based on that, we study the extreme case of the attainable constant 2 in Crouzeix's conjecture.

Necessary and sufficient conditions

We will present several necessary and sufficient conditions for a matrix A to be half-radial; for more details see [3, Theorem 9]. Half-radial matrices are closely related to the 2×2 Jordan block with the zero eigenvalue. Their field of values is always a disk with the radius $\|A\|/2$, centered at the origin. A half-radial matrix A has orthogonal maximum right and left singular subspaces; see [3, Lemma 6].

Half-radial matrices and Crouzeix's conjecture

Crouzeix's conjecture [2] states that

$$\|p(A)\| \leq 2 \max_{\zeta \in W(A)} |p(\zeta)|$$

holds for any square matrix A and any polynomial p.

In [4, p. 239], Greenbaum and Overton conjectured that if the equality holds in the above inequality for $A \in \mathbb{C}^{(n+1)\times(n+1)}$ and the polynomial $p(\zeta) = \zeta^n$, then A is unitarily similar to a scalar multiple of the $(n+1) \times (n+1)$ Crabb-Choi-Crouzeix matrix. Using results of Crabb [1] we show that their conjecture is true. We then generalize this result and present the structure of matrices satisfying for some $1 \leq k \leq n$ the equality

$$\|A^k\| = 2 \max_{\zeta \in W(A)} |\zeta^k| \,.$$

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New multiplicative formulae for general tridiagonal matrix inverse

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Keywords: matrix inversion, tridiagonal matrix, recursive algorithm, numerical stability

Even though in most problems involving matrix inverse the numerical computation of the actual inverse is usually not necessary (as the problem may be reformulated to solve a corresponding system of linear equations or a corresponding matrix equation), there seems to exist no computational system or numerical library which would miss a subroutine for numerical computation of the matrix inverse.

When using such a subroutine one could expect to obtain the most accurate result possible. Unfortunately, all numerical algorithms (known to the authors) for computing the matrix inverse suffer a *curse* that the larger of the residual errors, $\|AX - I\|$ and $\|XA - I\|$ (X denotes the computed inverse of a matrix A), can, in a pessimistic case, grow as fast as $\mathrm{cond}^2(A)$, where $\mathrm{cond}(A)$ is the condition number of A (we assume that A is not a triangular matrix).

In our presentation, we present the algorithm for inverting general tridiagonal matrices that overcomes the above curse, i.e. it computes the inverse for which both residual errors grow linearly with $\mathrm{cond}(A)$. In addition, the proposed algorithm has the smallest possible asymptotic complexity for the considered problem.

The proposed method is based on a careful selection of formulas for the elements of A^{-1} resulting from equations $AX = I = XA$. We will explain why not every choice of relations preserves all recursive properties and we will show how to choose the formulas in order to obtain an algorithm for which both residual errors are small.

Extensive numerical tests confirm very good numerical properties of this algorithm.

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